MATEMÁTICA

SOME REMARKS ON THE BINEUTRAL TOURNAMENTS

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SOME REMARKS ON THE BINEUTRAL TOURNAMENTS

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RESUMO Neste artigo apresentamos um estudo aprofundado sobre a classe dos Torneios Bineutrais, utilizando seus ciclos minimais, seus vertices neutrais e não neutrais. Como o refinamento de uma teoria prospectiva sobre os Grafos Derivados tem-se demonstrado muito util para a classificação de certas classes de torneios hamiltonianos, aplicamos esta nova abordagem à classe dos Torneios Bineutrais

Palavras-chave: Digrafos; Torneios Hamiltonianos; Ciclos Minimais; Caracteristica ciclica; Derived Graphs. Torneios Bineutrais.

ABSTRACT In this paper we throughly study the Bineutral Tournaments. We present all the main properties of these tournaments, using their minimal cycles, neutral and non-neutral vertices. Since the refinement of a prospective theory on associated derived digraphs has shown to be very useful for the sake of classifying several types of hamiltonian tournaments, in a quite beautiful way, we apply this new approach to the class of the Bineutral Tournaments

Keywords: Digraphs; Hamiltonian Tournaments; Minimal Cycles; Cyclic characteristic; Derived Graphs; Bineutral Tournaments.

1 Introduction

The bineutral tournaments form an important class of the hamiltonian tournaments. So in this paper we take a closer look at them.

In section 2 we summarize the concepts, notations and important results about digraphs and tournaments that we shall use in this paper. Many of those concepts arouse from a new approach to study hamiltonian tournaments from Demaria's Regular Homotopy Theory for Digraphs view point.

In section 3 we throughly study the class of the bineutral tournaments A_n which were introduced by J. W. Moon in 1966. We also show the importance f these tournaments in the characterization of certain classes of hamiltonian tournaments (e.g. Normal Tournaments).

In the last section we describe the new contruction of the Derived Graph associated to a hamiltonian tournament, presenting some of its main properties. Finally we give a complete description orf the derived graphs associated to the bineutral tournaments.

2 Notation and Auxiliary Results

The notations and definitions we shall use in this paper mainly follow [1]. In tis section most of the results we present can be found in [2], [3], [4], [6], [12], [13], and [14] and [16].

A digraph D is called a *tournament* if every pair of different vertices of D is joined by one and only one arc. A tournament T is called *hamiltonian* if it contains a spanning cycle, i.e. a cycle through all the vertices of T.

Let T be a tournament. If there is an arc from a vertex x to a vertex y in T, we say that x *dominates* y and denote it by $x \to y$. If A and B are two subtournaments of T and every vertex of A dominates each vertes of B, then we say that A *dominates* B and denote it by $A \to B$.

The *out-neighbourhood* $N^+(x)$ of a vertex x is the set of all vertices of T

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dominated by x. The *in-neighbourhood* $N^-(x)$ of a vertex x is the set of all vertices of T which dominate x.

The number of vertices in the out-neighbourhood (in-neighbourhood, respectively) of x is the *outdegree* $d^+(x)$ (*indegree* $d^-(x)$, respectively) of x. In case N is a subtournament of T, we shall denote by $d^+_N(x)$ and $d^-_N(x)$, the *outdgree* and the *indegree of* x *relative to* N, respectively.

If T is a tournament of *order* m we denote it by T_m, if T_m is hamiltonian we denote it by H_m.

By C_r usually we denote a *cycle* with r vertices, as well as the subtournament $\langle C_r \rangle$ spanned by its vertices. The singleton x, with $x \in T_m$, and the spanned subtournament $\langle x \rangle$ is simply denoted by x.

If C is a cycle in a tournament T and a vertex $x \in T - C$, we denote by $d^+_C(x)$ ($d^-_C(x)$, respectively) the outdegree $d^+_M(x)$ (indegree $d^-_M(x)$, respectively) relative to $M = \langle C \cup \{x\} \rangle$.

Tr_m is the *transitive tournament* of order m (that is, $x_i \to x_j \quad \Leftrightarrow \quad i < j$) and Tr^*_m its dual.

A vertex x in T_m *cones* a subtournament R if and only if $v \to R$ or $R \to v$ in T_m. Otherwise, we say that R is *non-coned* in T_m.

A subtournament S of T_m is an *e-component* of T_m (and its vertices are called *equivalent*) if S is coned by every vertex x in $T_m - S$. The whole tournament T_m and the single vertices are called *trivial e-components*.

Every tournament T_m can be partitioned in a certain number of disjoint e-components $S^1, \ldots, S^n$, which can be considered as the vertices ($w_1, \ldots, w_n$, respectively) of a tournament Q_n, so that T_m is the *composition* $Q_n(S^1, \ldots, S^n)$ of the e-components $S^1, \ldots, S^n$ with the *quotient* Q_n.

In other words, $T_m = S^1 \cup \ldots \cup S^n$ and $x \to y$ in T_m if, and only if, $x \to y$ in some S^i or $x \in S^j$, $y \in S^k$ and $w_j \to w_k$ (that is, $S^j \to S^k$) (see [12]).

Proposition 2.1. *Any quotient tournament Q_n of a tournament T_m is isomorphic to some subtournament of T_m.*

Proposition 2.2. *A tournament H_m is hamiltonian if, and only if, every one of its quotient tournaments is hamiltonian (or, equivalently, if and only if it has a hamiltonian quotient tournament).*

A tournament T_m is *simple* if it has no non-trivial e-component. That is, if $T_m = Q_n(S^1, \ldots, S^n)$, then $m = n$ or $n = 1$. If T_m is not simple, then we say it is *compound*. We say Q_n is a *simple quotient* of T_m if $T_m = Q_n(S^1, \ldots, S^n)$ and Q_n is simple.

Proposition 2.3. *Every tournament T_m, with $m \geq 2$, has exactly one simple quotient tournament Q_n (up to isomorphisms). Moreover:*
(a) if T_m is not hamiltonian, then $n = 2$;
(b) if T_m is hamiltonian, then $n \geq 3$ and the e-components which correspond to the simple quotient are uniquely determined.

The obvious homomorphism $p : T_m \to Q_n$ is called the *canonical projection*.

In [4], Burzio and Demaria introduced the concepts of *coned* and *non-coned* cycles in tournaments. Let H_m be a hamiltonian tournament. A non-coned cycle C of H_m is said to be *minimal non-coned* or, simply, *minimal*, if the hamiltonian subtournament $\langle C \rangle$ is non-coned but all its proper hamiltonian subtournaments are coned in H_m.

A *characteristic cycle* in H_m is a minimal cycle with the minimal lenght. This minimal lenght is called the *cyclic characteristic* of H_m, and we denote it by $cc(H_m)$.

The *cyclic difference* of H_m is the positive integer $\mathrm{cd}(H_m) = m - \mathrm{cc}(H_m)$.

In [4] it was proved that $2 \leq \mathrm{cd}(H_m) \leq m - 3$ or, equivalently, $3 \leq \mathrm{cc}(H_m) \leq m - 2$.

A vertex x of a hamiltonian tournament H_m is *neutral* if $H_m - x$ is hamiltonian. Otherwise, the vertex x is called *non-neutral*. By $\nu(H_m)$ ($\mu(H_m)$, respectively) we denote the number of all neutral (non-neutral, respectively) vertices of H_m. It is easy to see that

$$\begin{aligned} &\nu(H_m) + \mu(H_m) = m, \\ &2 \leq \nu(H_m) \leq m, \\ &0 \leq \mu(H_m) \leq m - 2\ . \end{aligned} \tag{1}$$

A tournament H_m is *normal* if it has a unique minimal cycle (namely, the characteristic one). Equivalently, H_m is normal if and only if $\mathrm{cd}(H_m) = \nu(H_m)$ (see [6]).

We have a characterization of the hamiltonian tournaments in terms of non-coned cycles, that was given by Burzio and Demaria in [4].

Proposition 2.4. *A tournament H_m, with $m \geq 5$, is hamiltonian if, and only if, there exists a non-coned n-cycle in H_m, with $3 \leq n \leq m - 2$.*

We observe that H_3 and H_4 also contain non-coned 3-cycles, but the condition $n \leq m - 2$ is not satisfied. We now present some important and useful (as we shall see in the applications) properties of the non-coned cycles, minimal cycles, neutral and non-neutral vertices.

Proposition 2.5. *Let H_m be a hamiltonian tournament. If C is a non-coned cycle in H_m, then the vertices in $H_m - C$ are all neutral vertices.*

Proof. See Proposition 3.2 in [14]. □

This proposition motivates the following definition (see [14]):

Definition 2.1. If C is a non-coned cycle in H_m, the set $P_C = H_m - C$ consists of neutral vertices of H_m, which are called *poles* of C.

Then it follows from the previous Proposition:

Proposition 2.6. *Let H_m be a hamiltonian tournament. If N (Q, respectively) is the subtournament of the neutral (non-neutral, respectively) vertices of H_m, then*

$N = \cup \{P_C \mid C \text{ non} - \text{coned cycle}\}$ *and* $Q = \cap \{V(C) \mid C \text{ non} - \text{coned cycle}\}$.

The next result describes the subtournament of the neutral vertices in terms of the minimal cycles (see Proposition 3.4 in [14]).

Proposition 2.7. *Let H_m be a hamiltonian tournament. A vertex x in H_m is neutral if, and only if, there exists a minimal cycle C in H_m, such that $x \in P_C$.*

Corollary 2.8. *Let H_m be a hamiltonian tournament. If N (Q, respectively) is the subtournament of the neutral (non-neutral, respectively) vertices of H_m, then $N = \cup \{P_C \mid C \text{ minimal cycle }\}$ and $Q = \cap \{V(C) \mid C \text{ minimal cycle }\}$.*

We also have (see Proposition 3.7 in [14]):

Proposition 2.9. *If $C_1, \ldots, C_r$ are non-coned cycles in a hamiltonian tournament H_m, then the subtournament $R = \langle C_1 \cup \ldots \cup C_r \rangle$ spanned by their vertices is hamiltonian.*

We now introduce the following definition:

Definition 2.2. Let T_m be a tournament of order m. If $T_m = Tr_n^*(S^{(1)}, \ldots, S^{(n)})$, with Tr_n^* being the dual transitive tournament of order n, and every component $S^{(i)}$ being a singleton or a hamiltonian subtournament, we say we have a *composition in strong components* of T_m.

We observe that the tournament H_m is hamiltonian if, and only if, H_m is the only strong component.

In the case $Q = \emptyset$, that is, in H_m all the vertices are neutral vertices, then $N = H_m$, hence obviously hamiltonian. On the other hand, if $Q \neq \emptyset$, the situation is not the same as we can see from the next result (see Theorem 3.8 in [14]).

Theorem 2.10. *Let H_m be a hamiltonian tournament. If there is at least one non-neutral vertex in H_m (that is, $Q \neq \emptyset$), then the subtournament N of the neutral vertices of H_m is not hamiltonian, or H_m is the composition of two singletons and a hamiltonian component H', with a 3-cycle as quotient.*

3 The Bineutral Tournaments

In 1966, J. W. Moon (see [19] proved the following:

Theorem 3.1. *The minimal number of $k-cycles$ in a hamiltonian tournament of order n is $n-k-1$.*

It was in this same article that he introduced A_n (with $n \geq 4$) the bineutral tournament of order n which has all the extremal properties, that is, for every k, with $4 \leq k \leq n$, A_n has exactly $n-k+1$ cycles of length k.

Later several researchers tried to characterize and enumerate the class of those tournaments having those extremal properties.

In 1970, R. J. Dougls (see [10] presented a characterization for the class D_n of all the hamiltonian tournaments having a unique$n-cycle$. Then in 1972, M. R. Garey (see [11]) enumerate the class of the Douglas tournaments proving that he number of elements in the class

D_n is equal to F_{2n-6} with $n \geq 4$, with F_i the $i-th$ Fibonacci number.

In 1975, M. Las Vergnas (see [17]) proved that $4 \leq k \leq n-1$ the tournament $A_n$is the unique hamiltonian tournament having $n-k+1$ k-cycles.

In 1990, M. Burzio and D. C. Demaria (see [5])presented a characterization for the class B_n of the hamiltonian tournaments of order n having the minimal number $n-2$ of 3-cycles. In this same article they proved that number of elements in the class B_n is equal to 2^{n-4}, with $n \geq 4$, and the number of tournaments in this class that are simple is exactly F_{n-4}.

In 1990, D. C. Demaria and J. C. S. Kiihl (ee [7] presented another structural characterization for the class D_n of hamiltonian tournaments of order n having a unique $n - cycle$. These tournaments are now called the *Douglas tournaments*.

In [8] D. C. Demaria and J. C. S. Kiihl using this new structural characterization for the class D_n and some variations of the Pascal triangle associated to some generalized Fibonacci number proved that he number of elements in the class of tournaments

D_n is equal to F_{2n-6}

with $n \geq 4$, getting the same result as Garey's in [11].

We recall that by Tr_n we denote the *transitive tournament*, such that $V(Tr_n) = \{t_1, t_2, \ldots, t_n\}$ and $t_i \to t_j$ if and only if $i < j$. We observe that this tournament is non hamiltonian. The vertex t_1 is called the *source* and the vertex t_n the *sink*.

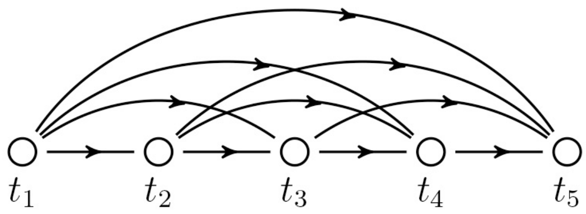

Figure 1: Tournament Tr_n for $n = 5$.

Let ATr_n denote the *almost transitive tournament*, which is obtained from Tr_n by replacing the arc $t_1 \to t_n$ by $t_n \to t_1$ and maintaining all the others. We observe that this tournament is hamiltonian. We can also see that all the 3-cycles: $t_1 \to t_i \to t_n \to t_1$, with $2 \leq i \leq n-1$ are all non-coned. In fact we have $cc(ATr_n) = 3$.

Remark 3.1. In Tr_n, with $n \geq 4$, if one of the arcs $t_i \to t_j$ is replaced by $t_j \to t_i$, with $i \neq 1$, the resulting tournament is still non hamiltonian. But there are some hamiltonian tournaments that can be obtained by reversing some of the arcs in Tr_n, as we shall see in the next example.

We recall here the definition of the bineutral tournament, which was given in the previous section.

Let A_n, with $n \geq 4$, denote the *bineutral tournament of order* n, that is the tournament such that $V(A_n) = \{t_1, t_2, \ldots, t_n\}$ and $t_i \to t_j$ if, and only if, $j < i - 1$ or $j = i + 1$.

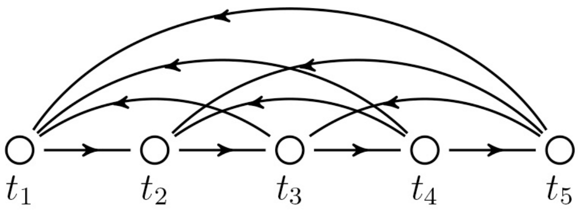

Figure 2: The bineutral A_n for $n = 5$.

Remark 3.2. It is easy to see that this hamiltonian tournament is also obtained from the transitive one, by reversing some of the arcs.

Remark 3.3. The tournament A_m, with $m \geq 4$, having vertex set $\{t_1, \ldots, t_m\}$ and such that $t_i \to t_j$ if and only if $j < i - 1$ or $j = i + 1$, is the only tournament with $\nu(A_m) = 2$. It is called the *bineutral tournament* of order m (see [19]). The subtournament spanned by $\{t_{n-1}, t_n, t_1, t_2\}$ is its maximal transitive subtournament, formed by consecutive vertices of the hamiltonian cycle. It is easy to see that $t_2 \to \cdots \to t_{n-1} \to t_2$ is its only minimal cycle (the *characteristic* one). Hence the bineutral tournament A_m is normal, with $\mathrm{cc}(A_m) = m - 2$, if and only if $m \geq 5$. The 3-cycle H_3 is normal. And the only hamiltonian tournament H_4 is not normal, for it has two minimal cycles and $\mathrm{cc}(A_4) = 3$. It is also known that A_5 is the only normal tournament of order 5.

It is known that A_n is the unique tournament having exactly two neutral vertices. Moreover, for $n \geq 5$, $\{a_{n-1}, a_n, a_1, a_2\}$ is its maximal transitive subtournament, formed by consecutive vertices of the hamiltonian cycle. It is easy to see that $a_2 \to \cdots \to a_{n-1} \to a_2$ is the only minimal cycle (the *characteristic* one). So that A_n is normal, having $cc(A_n) = n - 2$, if $n \geq 5$. We also observe that $cc(A_4) = 3$, with two minimal cycles (so that it is not normal). If we set A_3 to be the 3-cycle, then we also have $cc(A_3) = 3$, and it is also normal.

As we have said before, this new approach to study the tournaments, analysing them from the homotopical point of view, taking in consideration the cyclic characteristic, minimal cycles, neutral and non-neutral vertices, it yields many structural characterization theorems. We present now some of those results.

In 1989, Demaria and Gianella defined H_n to be a *normal tournament* if it is hamiltonian and has a unique characteristic cycle. In [6] they throughly studied this class of tournaments, which turned out to be very important in some structural characterization theorems for other classes of tournaments.

We present here some of the most important properties of the normal tournaments.

In [6] Demaria and Gianella have also shown that a normal tournament H_n $(n \geq 4)$ has as its characteristic cycle either the 3-cycle or a bineutral tournament A_k. In the same paper they have proved the following proposition, where by A_3 we denote the 3-cycle.

Proposition 3.2. *Let H_n be a normal tournament with cyclic characteristic k $(k \geq 3)$ and let A_k be its characteristic cycle. A pole z, associated to A_k, must have the following adjacencies with respect to A_k:*
1) $(a_{i+1}, a_{i+2}, \ldots, a_k) \to z \to (a_1, a_2, \ldots, a_i)$ $(1 \leq i \leq k-1)$.
2) $(a_i, a_{i+2}, a_{i+3}, \ldots, a_k) \to z \to (a_1, \ldots, a_{i-1}, a_{i+1})$ $(1 \leq i \leq k-1)$.

Definition 3.4. The pole z is called a *pole of kind* i *and class* 1 *or class* 2 (and denoted by x_i or y_i) if its adjacencies are given by the previous conditions 1) or 2), respectively.

The class of the normal tournaments is very important in the study of the hamiltonian tournaments, for instance, the class of the hamiltonian tournaments which have a unique n-cycle, which was characterized by Douglas (see [10]), can now be characterized in a different way as it is shown in:

Proposition 3.3. *Let H_n be a hamiltonian tournament with $cc(H_n) = k \geq 3$. H_n is a Douglas tournament if, and only if:*
1.1) H_n has as a simple quotient Q_m $(m \geq 5)$ such that:
a) Q_m is normal;
b) the subtournament of the poles in Q_m is transitive;
c) the poles of Q_m are all of class 1;
d) between two poles x_i and $x_j^{'}$ of Q_m of class 1, the following rule of adjacency holds: $x_i \to x_j^{'}$ implies $j \leq i+1$.
1.2) H_n can be constructed from Q_m by replacing all the vertices of Q_m, but the vertices $a_2, \ldots, a_{k-1}$ of its charcteristic cycle A_k, by some transitive tournament.
2) H_n is the composition of a singleton and two transitive tournaments with a 3-cycle.

Proof. See [8]. □

Later Demaria and Kiihl, using this characterization and the structural characterization of the normal tournaments given in [6] (by Demaria and Gianella), obtained the enumeration of the Douglas tournaments with a convenient variation of the Pascal triangle (see [9]).

4 Bineutral Tournaments and Associated Derived Graphs

In this section we shall construct the derived graph $\mathcal{G}_{A_n}$ associated with the bineutral tournament A_n.

First of all we shall describe how to construct the derived graph $\mathcal{G}_H$ associated with a hamiltonian tournament H.

Let H be any hamiltonian tournament. The *derived graph* (also called *associated 3-cycle digraph*) of H, or *associated derived graph* for short, will be defined as a directed graph $\mathcal{G}_H$ given as follows: the vertices of $\mathcal{G}_H$ are the 3-cycles of H, and the edges of $\mathcal{G}_H$ are the 4-cycles of H. That makes sense because any 4-cycle is isomorphic to H_4 - the only hamiltonian tournament of order 4 - and it contains precisely two 3-cycles.

To see how the edges of $\mathcal{G}_H$ are oriented consider C_1, C_2 two vertices of $\mathcal{G}_H$ joined by an edge (that is, two 3-cycles of H both of them contained in the same 4-cycle of H). Let $L = \langle C_1 \cup C_2 \rangle$, then C_1 and C_2 share a common edge, and each one of C_1, C_2 has a single vertex that is not shared with the other 3-cycle. Call it the *distinguished vertex* of that 3-cycle respect to L. We set

$$C_1 \to C_2 \quad \text{in } \mathcal{G}_H \tag{2}$$

if the distinguished vertex of C_1 precedes the distinguished vertex of C_2. Similarly, we set $C_1 \leftarrow C_2$ if the other adjacency relation holds between the distinguished vertices.

The first thing to notice about this conctruction is that the associated graph $\mathcal{G}_H$ obtained from a hamiltonian tournament H need not be a tournament. In general, many pairs of vertices of $\mathcal{G}_H$ will not be joined by edges. Further, it soon became clear to us that *non-isomorphic hamiltonian tournaments can have the same associated graphs.* Still, the refinement of a prospective theory on associated 3-cycle digraphs has shown to be very useful for the sake of classifying several types of hamiltonian tournaments in a quite beautiful way.

We give a brief presentation of the first definitions, examples and basic results, as well as some immediate applications, such as the contruction of the derived graphs associated with the bineutral tournaments.

Let H be any hamiltonian tournament with associated graph $\mathcal{G}_H$.

If $C, C' \in \mathcal{G}_H$ we write $C — C'$ in case C and C' are adjacent, so that $C — C' \Leftrightarrow (C \to C' \text{ or } C \leftarrow C')$. Otherwise we denote $C \not— C'$.

For an arbitrary graph G we denote by $\mathscr{V}(G)$ the set of vertices of G. If $C \in \mathcal{G}_H$ and $\mathscr{V}(C) = \{a, b, c\}$, one way to represent the 3-cycle C by its vertices and adjacencies is $C : a \to b \to c \to a$.

Let $C \in \mathcal{G}_H$ and $x \in H - C$. We say that x *cones* C if $x \to \mathscr{V}(C)$ or $x \leftarrow \mathscr{V}(C)$. The vertex $x \notin C$ *generates a 3-cycle* C' *with* C if there is $C' \in \mathcal{G}_H$ such that $C' — C$ and $x \in C'$. In this case $\mathscr{V}(C') = \{x, a, b\}$ where $a \to b$ is an edge of C, and we also say that x generates C' with $a \to b$. The first important result is

Proposition 4.1. *The vertex x generates a 3-cycle C' with C if and only if x does not cone C. Further, C' is the only 3-cycle generated by x and C.*

Returning to the class of the bineutral tournaments, we have following:

Proposition 4.2. *The bineutral tournament A_n, with $n \geq 3$ has $n-2$ 3-cycles. Namely:*

$$t_1 \to t_2 \to t_3 \to t_1,\ t_2 \to t_3 \to t_4 \to t_2,\ t_3 \to t_4 \to t_5 \to t_3,\ \ldots,\ t_{n-2} \to t_{n-1} \to t_n \to t_{n-2}.$$

Proof. It is obvious that they are 3-cycles. Let us show they are unique. If $t_i \to t_j \to t_k \to t_i$ is any 3-cycle in A_n, we can assume that $i = mini, j, k$. Since $k > i$ and $t_i \to t_j$, then the unique possibility is $j = i + 1$. But we must necessarily have that $k \geq j$ for $k > i$. Since we have that $k \neq j = i + 1$ and $t_j \to t_k$ then we must have $k = j + 1 + i + 2$. Therefore the 3-cycle is of the form $t_i \to t_{i+1} \to t_{i+2} \to ti$as we wanted to prove.

□

Let us denote by C_i the 3-cycle $t_i \to t_{i+1} \to t_{i+2} \to t_i$ in A_n.

We see that in the associated derived graph $\mathcal{G}_{A_n}$, the vertices determined by C_i and C_{i+1} are adjacent since they share in common the arc $t_{i+1} \to t_{i+2}$. Besides, since in A_n we have that $t_i \to t_{i+1}$ it follows that in $\mathcal{G}_{A_n}$ we have that $C_i \to C_{i+1}$.

Therefore we can conclude that, given the bineutral tournament A_n its associated derived graph $\mathcal{G}_{A_n}$ is the directed path:

$$C_1 \to C_2 \to \ldots \to C_{n-3} \to C_{n-2}.$$

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