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SOME REMARKS ON NON-RECONSTRUCTABLE TOURNAMENTS

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RESUMO Neste artigo apresentamos um estudo sobre parte do conjunto de torneios não-reconstruiiveis e algumas de suas propriedades combinatorias são deduzidas, levando a novas questões a respeito do problema da reconstrução de torneios

Palavras-chave: Digrafos; Torneios Hamiltonianos; Ciclos Minimais; Caracteristica ciclica; Torneios de Stockmeyer.

ABSTRACT Part of the whole set of already known non-reconstructable tournaments is considered and some of their combinatorial properties are deduced, leading to new questions related to the reconstruction problem for tournaments

Keywords: Digraphs; Hamiltonian Tournaments; Minimal Cycles; Cyclic characteristic; Stockmeyer Tournaments.

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1 Introduction and preliminaries.

It is well known that the reconstruction conjecture for digraphs and, in particular, for tournaments has been shown to be false by Stockmeyer [ST.1, ST.2]. Thence many questions in the reconstruction problem for tournaments still remain to be solved.

The main challange is, of course, to find a characterization, if any, of reconstructable tournaments and to this extent partial results may be of some interest provided they significatively enlarge either the class of reconstructable tournaments or the class of non-reconstructable ones. Also the reconstruction of combinatorial properties and invariants of tournaments may turn out to be useful.

Results of this kind may be found in [B.-P., D.M.-G., G1, G2, H.P., M., S.0, S.1, S.2, V.].

In the present paper we consider all the already, given pairs of non-reconstructable tournaments and investigate which of them satisfy certain combinatorial properties that seem to be related in some way to the reconstruction problem.

From the obtained results and from some remarks on well known classes of reconstructable tournaments new questions and problems arise and we formulate them in the concluding section.

We recall that pairs of non-reconstructable tournaments were found of order 3,4,5,6 (4 pairs), 8 (2 pairs), $2^n + 2^m, n \geq 3, m \geq 0$.

For a complete list of non-reconstructable tournaments of order $n \geq 8$ one should refer to [ST.0] and to [G.2] as well.

In the list given in [ST.0] one pair of order 6 is missing. In fact the tournaments M_6^1, M_6^2 of figure 1 in [G.2] which are also draff in the book of Moon [MO] on page 94 (row 3 column 2 and row 4 column 2, respectively), constitute a counterexample not listed in [B. -P., Stock 0]. This omission is surprising since Beineke and Parker apparently used the list of non-isomorphic tournaments of 6 vertices in [MO] and their default does not depend on an evident mistake in the book of Moon, where the tournament draft on row 1 column 1 of page 95 is isomorphic to the one pictured in row 4 column 2 page 93. The missing tournament in the list of [MO], with score vector (1, 2, 2, 3, 3, 4), can be obtained from the one in row 1 column 2 page 95 by changing the orientation of perimetral arcs of the square draft in the middle.

The counterexamples constructed by Stockmeyer in [ST.2] are described in the beginning of next section.

A compendium of the known classes of reconstructable tournaments can be found in [G.2].

For the definition of cards, hypomorphisms and the formulation of various reconstruction problems, that throughout this paper will be referred to vertex-reconstruction, we shall use the standard notation following [N.-W].

Now we recall some definitions and results we shall need later. For further details see [B.-R.,G2, K.-T.-Go.]. We denote by $T_n(H_n)$ a (hamiltonian) tournament with n vertices and by C_r, a cycle in T_n with r vertices as well as the subtournament with the same vertices; the same simbols will be used to denote the corresponding sets of vertices.

If $x, y \in T_n, x \to y$ means that x dominates y and we write $A \to B, A, B \subseteq T_n$, if every vertex of A dominates every vertex of B in T_n. A singleton will be usually identified with its only element. Tr_n denotes the standard transitive tournament with n vertices and $HR(m), m \geq 1$, the highly regular tournament whose vertices $v_1, \ldots, v_{2m+1}$ can be ordered in such a way that every vertex v_i dominates exactly the vertices with indices $i + 1, \ldots, i + m$ (mod 2m+1). We say that a vertex x cones a subtournament R in T_n if either $x \to R$ or $R \to x$ in T_n.

An e-component of T_n is a subtournament S which is coned by every vertex of $T_n - S$; the vertices of S are then said to be equivalent. Single vertices and T_n are trivial e-components.

We say a tournament T_n is simple if it has no non-trivial e-component; otherwise it is compound.

Every tournament T_n can be partitioned into e-components $S^1, \ldots, S^m$ other than T_n: the tournament Q_m with vertices $v_1, \ldots, v_m$ such that $v_i \to v_j$ iff $S^i \to S^j$ is a quotient of T_n. Then T_n is isomorphic to the composition tournament $Q_m(S^1, \ldots, S^m)$. Every tournament $T_n, n \geq 2$, has exactly one simple quotient Q_m: this is isomorphic to T_2 if T_n is not hamiltonian; otherwise it is determined by the unique partition of T_n into maximal e-components $S^1, \ldots, S^m, m \geq 3$, other than T_n and it is hamiltonian.

A vertex $x \in H_n$ is neutral if the relative card $H_n - x$ is hamiltonian. A tournament T_n is strongly 2-connected if all its cards are strong; hence the class of strongly 2-connected torunaments is closed under hypomorphisms.

A cycle C_r in a strong tournament H_n is minimal if it is not coned in H_n and every cycle $C_s, s < r$, whose vertices are in C_r is coned in H_n.

H_n is normal if it has just one minimal cycle. The cyclic characteristic of H_n is the minimal length of the non-coned cycles.

Hamiltonian simply disconnected tournaments H_n can be characterized by either one of the following equivalent conditions: (i) a 3-cycle is coned in H_n iff it is included in a non trivial e-component; (ii) the simple quotient of H_n is the highly regular tournament $HR(m)$ for some $m \geq 1$.

2 Cyclic characteristic on non-reconstructable tournaments.

For each non-negative integer n, let A_n be a tournament with vertices $v_i, 1 \leq i \leq 2^n$, and dominance relations given by

$$\begin{aligned} v_i \to v_j \quad & \text{iff} \quad \text{odd } (j-i) \equiv 1 (\text{mod } 4), \quad \text{for } \ i \neq j. \\ & \text{iff} \quad \exists a \in T, r \geq 0 \ \text{ such that } \ j = i + (4a+1)2^r. \end{aligned}$$

Let m, n be non-negative integers, such that $0 \leq m < n$. Consider the tournament $D_{n,m}$ whose vertices are those of $A_n, v_1, \ldots, v_{2^n}$, and those of A_m, $v_{2^n+1}, \ldots, v_{2^n+2^m}$ (distinct vertices correspond to distinct indices). The dominance relations among the first 2^n vertices remain the same as in A_n, and among the last 2^m vertices are the same as in A_m. The even (odd) vertices in A_n dominate the even (odd) vertices in A_m.

We also consider the tournament $D^*_{n,m}$, having the same vertex-set as $D_{n,m}$, reverting the dominance between the vertices in A_n and the vertices in A_m

Observe, that $D_{n,m}$ and $D^*_{n,m}$ are both hamiltonian tournaments, for $n \geq 2$. These tournaments, with a different notation, were constructed in [Stock 2] and, for the case $0 \leq m \leq 1$, in [Stock 1]. In those papers, Stockmeyer has proved the following results.

Proposition 2.1. *For all positive integers m, n, such that $0 \leq m < n$ we have:*

(i) $m = 0 \Rightarrow D_{n,m}$ and $D^*_{n,m}$ are self-complementary;

(ii) $m \geq 1 \Rightarrow D_{n,m}$ and $D^*_{n,m}$ are complements of each other;

(iii) $D_{n,m}$ and $D^*_{n,m}$ are non-isomorphic.

(iv) $D_{n,m}$ and $D^*_{n,m}$ have the same cards, that is, they are hypomorphic. Moreover, the card $D_{n,m} - v_{2^n+k}$, $1 - 2^n \leq k \leq 2^m$, is isomorphic to the card $D^*_{n,m} - v_{k'}$, where $k' \equiv 2^n + 2^m + 1 - k$ (mod $2^n + 2^m$). □

We say a vertex in $A_n, D_{n,m}$ or $D^*_{n,m}$ is even (odd) if its index (in the description given above) is even (odd).

Lemma 2.1. *Let $n \geq 2$, then we have:*

(i) There exist non-coned 3-cycles in A_n, with vertices that are not all even neither all odd, if and only if $n \leq 3$;

(ii) The vertices v_1, v_2, v_3, v_4 form a non-coned 4-cycle in A_n.

Proof. (i) For any considered 3-cycle in A_n it is possible do find an (unique) ordering of its vertices $(v_i \to v_j \to v_k)$ such that $\exists r \geq 0, \exists \alpha, \beta \in \mathbb{Z}, \alpha + \beta$ odd, so that

$$\begin{cases} i = k + (4\beta + 1)2^r \\ j = k - (4\alpha + 1)2^r. \end{cases}$$

We observe that for all k, with $1 \leq k \leq 2^n$, there exists such a 3-cycle iff $n \geq 3$. If $r > 0$, then the three vertices are all even or all odd.

Let $r = 0$ and denote by $C(ijk)$ the associated cycle; in this case $i \equiv j \not\equiv k \pmod 2$.

A vertex v_x exists which dominates $C(ijk)$ iff for some $1 \leq x \leq 2^n$ and for some $x_i, x_j, x_k \in \mathbb{Z}$ the following conditions are satisfied

$$\begin{cases} x = i - 2(4x_i + 1) \\ x = j - 4(4x_j + 1) \\ x = k - (4x_k + 1). \end{cases}$$

Hence such a vertex v_x exists iff $\exists x_j \in \mathbb{Z}$ such that $1 \leq j - 4(4x_j + 1) \leq 2^n$, and this happens iff j satisfies one of the conditions $5 \leq j \leq 2^n$ or $1 \leq j \leq 2^n - 12$.

Every index $1 \leq j \leq 2^n$ satisfies such a condition iff $n \geq 4$. Therefore in A_n, with $n \geq 4$, every 3-cycle with vertices which are not all even nor all odd it is coned.

Similarly, there exists a vertex v_y which is dominated by $C(ijk)$ iff $\exists\, 1 \leq y \leq 2^n$ and $\exists\, y_i, y_j, y_k \in \mathbb{Z}$ such that

$$\begin{cases} y = i + 4(4y_i + 1) \\ y = j + 2(4y_j + 1) \\ y = k + 4y_k + 1. \end{cases}$$

Hence such a vertex v_y exists iff $\exists y_i \in \mathbb{Z}$ such that $1 \leq i + 4(4y_i + 1) \leq 2^n$, and this happens iff i satisfies one of the conditions $1 \leq i \leq 2^n - 4$ or $13 \leq i \leq 2^n$. In this case, we also have that every index $1 \leq i \leq 2^n$ satisfies one of these conditions iff $n \geq 4$.

If $n = 2$, the only 3-cycles in $A_2, C(241)$ and $C(134)$, are non-coned and their vertices are not all even nor all odd.

If $n = 3$, the non-coned 3-cycles in A_3 having vertices which are not all even nor all odd are $C(712), C(823), C(716), C(827)$.

(ii) Excluding the case $n = 2$, which is trivial, with the notation introduced above, one may easily verify that if $C(241) \to v_y$ and $C(134) \to v_y$, then it must exist $t, t' \in \mathbb{Z}$ such that

$$\begin{cases} y = & 5 + 4t \\ & \text{and} \\ y = & 4 + 4t' \end{cases}$$

but this is impossible.

Similarly, $v_x \to v_i$, for $i = 1, \ldots, 4$, implies $\exists\, s, s' \in \mathbb{Z}$ such that

$$\begin{cases} x = & 3 - 4s \\ & \text{and} \\ x = & 2 - 4s' \end{cases}$$

which is also impossible.

□

Lemma 2.2. *Let $1 \leq p, p', d, d' \leq 2^n, n \geq 4$, with p, p' even and d, d' odd. Then we have in A_n*

(i) $v_p \to v_d \Rightarrow \exists 1 \leq \overline{d}, \overline{p} \leq 2^n, \overline{d}$ odd, $\overline{p}$ even, such that $v_{\overline{d}} \to \{v_p, v_d\} \to v_{\overline{p}}$;

(ii) $v_{d'} \to v_{p'} \Rightarrow \exists 1 \leq \tilde{p}, \tilde{d} \leq 2^n, \tilde{p}$ even, $\tilde{d}$ odd, such that $v_{\tilde{p}} \to \{v_{d'}, v_{p'}\} \to v_{\tilde{d}}$.

Proof. For every fixed even index $1 \leq p \leq 2^n$, all the arcs from v_p to an odd vertex are obtained in correspondence to the indices $1 \leq d \leq 2^n$ obtainable by the relation

$$d = p + 4k + 1, \qquad \text{for some} \quad k \in \mathbb{Z}.$$

If we consider any $\alpha, \beta \in \mathbb{Z}$ such that

$$\begin{cases} 1 \le p - 4\alpha - 1 \le 2^n \\ \alpha + k \text{ even} \end{cases} \quad \text{and} \quad \begin{cases} 1 \le d + 4\beta + 1 \le 2^n \\ \beta + k \quad \text{even} \end{cases}$$

the odd index $\overline{d} = p - 4\alpha - 1$ determines a vertex which dominates $\{v_p, v_d\}$ and the even index $\overline{p} = d + 4\beta + 1$ determines a vertex which is dominated by $\{v_p, v_d\}$.

Similarly, all the arcs from an odd vertex to an even vertex can be obtained by fixing an odd integer d' and determining an even integer satisfying

$$p' = d' + 4h + 1, \text{ for some } h \in \mathbb{Z}.$$

Any solutions $i, j \in \mathbb{Z}$ for

$$\begin{cases} 1 \le d' - 4i - 1 \le 2^n \\ i + h \text{ even} \end{cases} \quad \text{and} \quad \begin{cases} 1 \le p' + 4j + 1 \le 2^n \\ j + h \quad \text{even} \end{cases}$$

determine, by taking

$$\tilde{p} = d' - 4i - 1, \text{ and } \tilde{d} = p' + 4j + 1$$

the required indices in the second implication.

□

Proposition 2.2. *Let $n \ge 2$ and $0 \le m < n$.*
Then

$$\{n, m\} \cap \{1, 2, 3\} \ne \phi \iff cc(D_{m,n}) = 3,$$
$$\{n, m\} \cap \{1, 2, 3\} = \phi \iff cc(D_{n,m}) = 4.$$

Proof. Four cases have to be considered:

$$m = 0 \quad \text{and} \quad 2 \le n \le 3. \tag{1}$$

By lemma 2.2 there exist non-coned cycles in A_n with vertices which are not all even nor all odd; such cycles cannot be coned by v_{2^n+1} either. Therefore, $cc(D_{n,0}) = 3$.

$$m = 0 \quad \text{and} \quad n \ge 4. \tag{2}$$

The cycles in A_n which are non-coned in A_n their vertices are all even or all odd; then by lemma 2.2, they are coned by v_{2^n+1}. On the other hand, any 3-cycle having v_{2^n+1} contains an arc in A_n from an even to an odd vertex; then by lemma 2.3, such vertices certainly are dominated by an odd vertex which, obviously, dominates the entire cycle. From lemma 2.2 (ii) it follows that $cc(D_{n,0}) = 4$.

$$1 \le m \le 3. \tag{3}$$

From lemma 2.2 (i) it follows that in $D_{n,m}$ there exist non-coned 3-cycles with vertices in A_m, if $2 \le m \le 3$. If $m = 1$. Consider any odd vertex v_d in A_n, it is clear the 3-cycle through v_d and vertices in A_m is non-coned in $D_{n,m}$. Therefore $cc(D_{n,m}) = 3$, in any case.

$$m \ge 4. \tag{4}$$

From lemma 2.2 it follows that every 3-cycle contained in A_n or in A_m is coned. If C is a 3-cycle with an odd vertex in A_m and an arc $v_p \to v_d$ (with p even and d odd, obviously) in A_n, then by lemma 2.3 there exists an odd vertex in A_n dominating the entire cycle C.

Considering all the other analogous cases we conclude that every 3-cycle is coned. Again by lemma 2.2, it follows that $cc(D_{m,n}) = 4$.

□

Corollary 2.1. *For all intergers m, n, with $0 \leq m < m$, we have*

$$cc(D^*_{n,m}) = cc(D_{n,m}).$$

Proof. If $m \geq 1$, the result follows from Proposition 2.1 (ii).

If $m = 0$ and $n \leq 3$, by lemma 2.2 there exists a non-coned 3-cycle in $D^*_{n,1}$, hence $cc(D^*_{n,1}) = 3 = cc(D_{n,1})$.

Let us consider then $n \geq 4$. From lemma 2.2, it follows that every 3-cycle in A_n is coned in $D^*_{n,1}$. Let C be a 3-cycle through v_{2^n+1}; such a cycle contains an arc $v_{d'} \to v_{p'}$, with d' odd and p' even, whose vertices (by lemma 2.3) are dominated by an even vertex in A_n, which obviously dominates the entire cycle C. Therefore by lemma 2.2 (ii) it follows that $cc(D^*_{n,0}) = 4 = cc(D_{n,0})$. □

Remark 2.1. (i) All the known pairs of hypomorphic tournaments which are not isomorphic, of order 5, 6, 8, which are not included in corollary 2.5 consist of tournaments having the same cyclic characteristic, namely 3.

(ii) The counterexamples of order 3 and 4 consist of pairs of tournaments in which only one is hamiltonian.

3 Other properties of non-reconstructable tournaments

Proposition 3.1. *Let m, n be nonnegative integers, such that $0 \leq m < n$ and $p = 2^n + 2^m \geq 6$. Then $D_{n,m}$ and $D^*_{n,m}$ are strongly 2-connected tournaments.*

Proof. First of all observe that for $n \geq 3$ every vertex v_i dominates v_{i+1}, v_{i+2} (where $i+1$ and $i+2$ are reduced mod p). For $n = 2$, this is also true for v_1, v_2 and, we have $v_i \to v_{i+1}$, for all $1 \leq i \leq 4$.

Let $v \in D_{n,m}$. If $v = v_{2^n+j}$ in A_m, A_n is a cycle (observe that $n \geq 2$) which is not coned by any vertex in A_m, hence it can be extended to a hamiltonian cycle in the card $D_{n,m} - v$. If $v = v_i$ in A_n and $n \geq 3, A_n - v$ is a cycle which is not coned by any vertex in A_m, hence it can be extended to a hamiltonian cycle in the card $D_{n,m} - v$.

On the other hand let us suppose $n = 2$, and hence $m = 1$. In $A_n - v$ there exists an odd vertex such that with v_5 and v_6 they form a cycle which is non-coned in the card $D_{2,1} - v$, which is therefore hamiltonian.

Now it is a trivial consequence of Proposition 2,1 (iv) that every card in $D^*_{n,n}$, satisfying the hypothesis, it is also hamiltonian. □

Remark 3.1. The non reconstructable tournaments of order 5 and 6 which are distinct of $D_{2.1}$ and $D^*_{2,1}$ are the only known tournaments which are not strongly 2-connected.

Proposition 3.2. *Let m, n be non negative intergers such that $0 \leq m < n$ and $p = 2^n + 2^m \geq 6$.*
Then $D_{n,m}$ and $D^*_{n,m}$ are simple tournaments.

Proof. Let X be an e-component of $D_{n,m}$ and let $v_x \neq v_y$, be two distinct vertices in X.

(1) Let us suppose $1 \leq x, y \leq 2^n$

If $x \equiv y \pmod 2$, then the vertex relative to $z = x + \dfrac{y-x}{2}$ does not cone $\{v_x.v_y\}$, so $v_z \in X$. We can assume, without loss of generality, $z \not\equiv x \pmod 2$, changing y if necessary.

It is easy to see that

$$\begin{cases} x - t = (4a+1)2^{\alpha} \\ y - t = (4b+1)2^{\beta} \\ z - t = (4c+1)2^{\gamma} \end{cases} \qquad 1 \leq t \leq 2^n$$

it does not admit any integer solution, that is, there is no t, with $1 \leq t \leq 2^n$, such that $v_t \to \{v_x, v_y, v_z\}$.

Similarly, there is no t, with $1 \leq t \leq 2^n$. Such that $\{v_x, v_y, v_z\} \to v_t$.

In conclusion we have $v_t \in X$, for all $1 \leq t \leq 2^n$, that is, $A_n \subseteq X$. Since none of the vertices in A_m cones A_n, it follows that $X = D_{n,m}$.

If $x \not\equiv y \pmod 2$ then $A_m \subseteq X$. Then if $m \geq 1$, it follows that $A_n \subseteq X$, that is, $X = D_{n,m}$. On the other hand, if $m = 0$, hence $n \geq 3$, assume that y is even. If $y + 1 \not\equiv x \pmod 2$, then $v_y \to v_{y+1} \to v_{2^n+1}$, so that $v_{y+1} \in X$ and, as before, $X = D_{n,0}$. If $y + 1 \equiv x \pmod 2$ and $y \geq 3$ (otherwise, $x \leq n - 2$) then $v_x \to v_{y-2} \to v_y$ (otherwise, $v_x \to v_{x+2} \to v_y$), and hence $v_{y-2} \in X$ (otherwise, $v_{x-2} \in X$) and, as before it yields in any case $X = D_{n,0}$.

(2) If we suppose $2^n + 1 \leq x, y \leq 2^n + 2^m$ (with $m > 0$ and $n \geq 2$, of course). We proceed as in the previous case and without any exception we get $X = D_{n,m}$.

(3) Finally, let us suppose $1 \leq x \leq 2^n$ and $2^n + 1 \leq y \leq 2^n + 2^m$.

If $x \equiv y \pmod 2$, then $v_x \to v_{x+2} \to v_y$, for $x \leq 2^n - 2$. Otherwise $x \geq 2$ and hence $v_y \to v_{x-1} \to v_x$. In any case there exist two disinct vertices of A_n in X. Therefore, as it was shown before, we have that $X = D_{n,m}$.

If $x \not\equiv y \pmod 2$, then $v_x \to v_{x+1} \to v_y$, for $x \leq 2^n - 1$. Otherwise $x \geq 3$ and hence $v_y \to v_{x-2} \to v_x$. In any case there exist two distinct vertices of A_n in X and again $X = D_{n,m}$. □

Remark 3.2. (i) All known non-reconstructable tournaments are simple, with the only exceptions: $D_{2,0}$, of order 5, and $C_3(T_1, T_2, Tr_3), C_3(T_1, Tr_3, T_2)$, of order 6.

(ii) For $n \geq 6$, the known pairs of non-constructable tournaments are formed by tournaments which are either both simple or both compound.

(iii) The simple quotient of every known non-reconstructable hamiltonian tournament is a non-reconstructable tournament.

4 Concluding remarks and problems.

The results given in section 2 suggest that the cyclic characteristic of hamiltonian tornaments is in some way related to reconstructability problems. Indeed it is not clear if some kind of direct link exists between reconstructable hamiltonian tournaments and their cycle characteristic, but actually, no non-reconstructable tournament H is known with $cc(H) > 4$.

On the other hand we believe that the more evident trend that hypormophic strong tournaments have the same cyclic characteristic in the considered known situations may be extended to all hamiltonian tournaments, so that we conjecture that the cyclic characteristic of any hamiltonian tournament is reconstructable.

We note that the class $\mathcal{H}^2$ of strongly 2-connected tournaments is closed under hypomorphisms, i.e. every tournament hypomorphic to any given strongly 2-connected one is also strongly 2-coonected, since its cards all are hamiltonian. The same is true for the class $\mathcal{H} - \mathcal{H}^2$ of hamiltonian non-strong 2-connected tournaments.

The results of section 3 show that surely, we cannot say that the elements of $\mathcal{H}^2$ can be reconstructed. As for the $\mathcal{H} - \mathcal{H}^2$ one could ask whether it has non-reconstructable tournaments with more than 7 vertices or if every element $H_n \in \mathcal{H} - \mathcal{H}^2, n \geq 7$, can be reconstruceted from its cards.

Other questions arise from results of section 3 about the simple quotient of a tournament: Can any tournament be reconstructed provided its simple quotient is reconstructable, or, equivalently, is it true that the simple quotient of every non-reconstructable tournament is not reconstructable?

Such a problem becomes more intriguing if we look at the already given classes of reconstructable torunaments. In fact the reconstructable tournaments exibited in [H.-P.] are exactly those having as simple quotient the tournament with 2 vertices which is, of course reconstructable. The class of simply disconnected tournaments, that have been reconstrcutable in [G.1, V.], contains all tournaments whose simple quotient is the highly regular tournament $HR(m)$, $m \geq 2$, which is reconstructable. Eventually in [G2] all tournaments are reconstructable whose simple quotient belongs to the class of reconstructable tournaments, namely the normal tournaments of order $n \leq 4$, considered in [D.M.-G.].

We further remark that for every $n \geq 6$ and for every pair of hypomorphic tournaments H_n, H_n^* already considered we have that H_n is simple if and only if H_n^* is simple. Consequently we ask wether "being simple" is a hypomorphical property for all tournaments.

We conclude our remarks by noting that the tournaments M_6^1 and M_6^2 of figure 1 in [G2], that constitute the missing pair in the list of counterexamples in [B.P., St.O], are converse of each other, which confirms the trend already remarked in [B.-P., St.0] for non-reconstructable tournaments of even order.

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