Matemática

# 6 - TOURNAMENTS HAVING A MINIMAL CYCLE OF LENGTH FOUR


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# 6-Tournaments having a minimal cycle of length four

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**Resumo:** Neste artigo apresentamos um estudo, usando os conceitos de ciclos minimais, vértices neutrais e não neutrais, dos torneios hamiltonianos de ordem 6 que possuem um ciclo minimal de comprimento 4. Mostramos que há 5 desses torneios de ordem 6, mas somente 1 tem o número maximal de vértices não neutrais.
**Palavras-chave:** Digrafos; Torneios Hamiltonianos; Ciclos Minimais; Vértices Neutrais e Não-neutrais; Torneios de Douglas e de Moon.

**Abstract:** In this paper we thoroughly study, using the concepts of minimal cycles, neutral and non-neutral vertices, the hamiltonian 6-tournaments having a minimal cycle of length four. We show there are five of those 6-tournaments, but only one has the maximal number of non-neutral vertices.
**Keywords:** Digraphs; Hamiltonian Tournaments; Minimal Cycles; Neutral and Non-neutral Vertices; Douglas and Moon Tournaments.

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# 1 Introduction

In a hamiltonian tournament $H_n$, a vertex $x$ is called a *neutral vertex* if $H_n - x$ is still a hamiltonian tournament; otherwise, the vertex $x$ is said to be *non-neutral*. These two concepts are very important in order to obtain structural characterization for some classes of hamiltonian tournaments, in terms of their *non-coned cycles* and *minimal cycles*. Several new results have already been obtained following this approach see [9, 10, 11]. In [13] we throughly have studied the 5-tournaments. In this paper we analyse the special case of the 6-tournaments having a minimal cycle of length 4.

In section 2, we present the basic concepts and some interesting results about tournaments and their quotients. We also recall the concepts of *non-coned cycles, minimal cycles, neutral* and *non-neutral vertices*, and some results stablishing relation among them and their main properties. We follow the notation and the definitions presented in [1] and [13]. We recall that this approach originates in the use of the Regular Homotopy Theory, introduced by Davide C. Demaria (see [3, 4, 5, 6, 7]). We shall remark the Regular Homotopy Theory for Digraphs is, in our opinion, the most correct and natural homotopy theory that should be used, since the regular maps are pre-continuous maps as it was shown in [12].

In section 3 we thoroughly study the 6-tournaments having minimal cycles of length 4. The classification of those tournaments employs the standard theory on the topology of graphs, as well as the concepts of coned and non-coned cycles. We also devised a new structure, the *associated graph of 3-cycles* of a hamiltonian tournament, that helps on distinguishing a couple of different hamiltonian tournaments where the usual topological characters coincide. This new structure will be explored in a forthcoming paper.

As a consequence of Lemmas 3.2, 3.3 and Remark 3.4, as well as the discussion in section 3, we are able to prove Theorem 3.6. This is the main theorem of this work, and states a classification for the hamiltonian 6-tournaments having minimal cycles of length four in terms of their number of minimal 3-cycles and 4-cycles and also the number of neutral vertices.

# 2 Tournaments, quotients, minimal cycles, neutral and non-neutral vertices

Let $T$ be a tournament. If there is an arc from a vertex $x$ to a vertex $y$ in $T$, we say that $x$ *dominates* $y$ and denote it by $x \to y$. If $A$ and $B$ are two subtournaments of $T$ and every vertex of $A$ dominates each vertes of $B$, then we say that $A$ *dominates* $B$ and denote it by $A \to B$.

The *out-neighbourhood* $N^+(x)$ of a vertex $x$ is the set of all vertices of $T$ dominated by $x$. The *in-neighbourhood* $N^-(x)$ of a vertex $x$ is the set of all vertices of $T$ which dominate $x$. The *neighbourhood* $N(x)$ is the union $N^+(x) \cup \{x\} \cup N^-(x)$.

The number of vertices in the out-neighbourhood (in-neighbourhood, respectively) of $x$ is the *outdegree* $d^+(x)$ (*indegree* $d^-(x)$, respectively) of $x$. In case $N$ is a subtournament of $T$, we shall denote by $d^+_N(x)$ and $d^-_N(x)$, the *outdegree* and the *indegree of $x$ relative to* $N$, respectively.

If $T$ is a tournament of *order* $m$ we denote it by $T_m$, if $T_m$ is hamiltonian we denote it by $H_m$. By $C_r$ usually we denote a *cycle*, $C_r : x_1 \to \cdots \to x_r \to x_1$, with $r$ vertices, as well as the subtournament $\langle C_r \rangle$ spanned by its vertices. The singleton $x$, with $x \in T_m$, and the spanned subtournament $\langle x \rangle$ is simply denoted by $x$.

If $C$ is a cycle in a tournament $T$ and a vertex $x \in T - C$, we denote by $d^+_C(x)$ ($d^-_C(x)$, respectively) the outdegree $d^+_M(x)$ (indegree $d^-_M(x)$, respectively) relative to $M = \langle C \cup \{x\} \rangle$.

$Tr_m$ is the *transitive tournament* of order $m$ (that is, $x_i \to x_j \iff i < j$) and $Tr^*_m$ its dual. A vertex $x$ in $T_m$ *cones* a subtournament $R$ if and only if $v \to R$ or $R \to v$ in $T_m$. Otherwise, we say that $R$ is *non-coned* in $T_m$.

A subtournament $S$ of $T_m$ is an *e-component* of $T_m$ (and its vertices are called *equivalent*) if $S$ is coned by every vertex $x$ in $T_m - S$. The whole tournament $T_m$ and the single vertices are called *trivial e-components.*

Every tournament $T_m$ can be partitioned into disjoint e-components $S^1, \ldots, S^n$, which can be considered as the vertices ($w_1, \ldots, w_n$, respectively) of a tournament $Q_n$, so that $T_m$ is the *composition*

$Q_n(S^1,\dots,S^n)$ of the e-components $S^1,\dots,S^n$ with the *quotient* $Q_n$. In other words, $T_m = S^1\cup\ldots\cup S^n$ and $x\to y$ in $T_m$ if, and only if, $x\to y$ in some $S^i$ or $x\in S^j$, $y\in S^k$ and $w_j\to w_k$ (that is, $S^j\to S^k$) (see [2]).

**Proposition 2.1.** *Any quotient tournament $Q_n$ of a tournament $T_m$ is isomorphic to some subtournament of $T_m$.* □

**Proposition 2.2.** *A tournament $H_m$ is hamiltonian if, and only if, every one of its quotient tournaments is hamiltonian (or, equivalently, if and only if it has a hamiltonian quotient tournament).*

A tournament $T_m$ is *simple* if it has no non-trivial e-component. That is, if $Q_n(S^1,\dots,S^n)$, then $m=n$ or $n=1$. If $T_m$ is not simple, then we say it is *compound*. We say $Q_n$ is a *simple quotient* of $T_m$ if $T_m = Q_n(S^1,\dots,S^n)$ and $Q_n$ is simple.

**Proposition 2.3.** *Every tournament $T_m$, with $m\geq 2$, has exactly one simple quotient tournament $Q_n$ (up to isomorphisms). Moreover:*
*(a) if $T_m$ is not hamiltonian, then $n=2$;*
*(b) if $T_m$ is hamiltonian, then $n\geq 3$ and the e-components which correspond to the simple quotient are uniquely determined.*

The obvious homomorphism $p: T_m\to Q_n$ is called the *canonical projection*.

In [5], Burzio and Demaria introduced the concepts of *coned* and *non-coned* cycles in tournaments. Let $H_m$ be a hamiltonian tournament. A non-coned cycle $C$ of $H_m$ is said to be *minimal non-coned* or, simply, *minimal*, if the hamiltonian subtournament $\langle C\rangle$ is non-coned but all its proper hamiltonian subtournaments are coned in $H_m$.

A *characteristic cycle* in $H_m$ is a minimal cycle with the minimal lenght. This minimal lenght is called the *cyclic characteristic* of $H_m$, and we denote it by $cc(H_m)$. The *cyclic difference* of $H_m$ is the positive integer $\mathrm{cd}(H_m) = m - \mathrm{cc}(H_m)$. In [5] it was proved that $2\leq \mathrm{cd}(H_m)\leq m-3$ or, equivalently, $3\leq \mathrm{cc}(H_m)\leq m-2$.

A vertex $x$ of a hamiltonian tournament $H_m$ is *neutral* if $H_m - x$ is hamiltonian. Otherwise, the vertex $x$ is called *non-neutral*. By $\nu(H_m)$ ($\mu(H_m)$, respectively) we denote the number of all neutral (non-neutral, respectively) vertices of $H_m$. It is easy to see that

$$\begin{aligned} &\nu(H_m)+\mu(H_m)=m,\\ &2\leq \nu(H_m)\leq m,\\ &0\leq \mu(H_m)\leq m-2\ . \end{aligned} \tag{1}$$

A tournament $H_m$ is *normal* if it has a unique minimal cycle (namely, the characteristic one). Equivalently, $H_m$ is normal if and only if $\mathrm{cd}(H_m)=\nu(H_m)$ (see [8]).

**Remark 2.4.** The tournament $A_m$, with $m\geq 4$, having vertex set $\{a_1,\dots,a_m\}$ and such that $a_i\to a_j$ if and only if $j<i-1$ or $j=i+1$, is the only tournament with $\nu(A_m)=2$. It is called the *bineutral tournament* of order $m$ (see [5]). The subtournament spanned by $\{a_{n-1},a_n,a_1,a_2\}$ is its maximal transitive subtournament, formed by consecutive vertices of the hamiltonian cycle. It is easy to see that $a_2\to\cdots\to a_{n-1}\to a_2$ is its only minimal cycle (the *characteristic* one). Hence the bineutral tournament $A_m$ is normal, with $\mathrm{cc}(A_m)=m-2$, if and only if $m\geq 5$. The 3-cycle $H_3$ is normal. And the only hamiltonian tournament $H_4$ is not normal, for it has two minimal cycles and $\mathrm{cc}(A_4)=3$. It is also known that $A_5$ is the only normal tournament of order 5.

We have a characterization of the hamiltonian tournaments in terms of non-coned cycles, that was given by Burzio and Demaria in [6].

**Proposition 2.5.** *A tournament $H_m$, with $m\geq 5$, is hamiltonian if, and only if, there exists a non-coned $n$-cycle in $H_m$, with $3\leq n\leq m-2$.*

We observe that $H_3$ and $H_4$ also contain non-coned 3-cycles, but the condition $n\leq m-2$ is not satisfied. We now present some important and useful (as we shall see in the applications) properties of the non-coned cycles, minimal cycles, neutral and non-neutral vertices.

**Proposition 2.6.** *Let $H_m$ be a hamiltonian tournament. If $C$ is a non-coned cycle in $H_m$, then the vertices in $H_m - C$ are all neutral vertices.*

*Proof.* See [13]. □

This proposition motivates the following definition ([8]):

**Definition 2.7.** If $C$ is a non-coned cycle in $H_m$, the set $P_C = H_m - C$ consists of neutral vertices of $H_m$, which are called *poles* of $C$.

**Proposition 2.8.** *Let $H_m$ be a hamiltonian tournament. If $N$ ($Q$, respectively) is the subtournament of the neutral (non-neutral, respectively) vertices of $H_m$, then $N = \cup\,\{P_C \mid C \text{ non} - \text{coned cycle}\}$ and $Q = \cap\,\{V(C) \mid C \text{ non} - \text{coned cycle}\}$.*

*Proof.* It follows immediately from the previous proposition. □

We shall now describe the subtournament of the neutral vertices in terms of the minimal cycles.

**Proposition 2.9.** *Let $H_m$ be a hamiltonian tournament. A vertex $x$ in $H_m$ is neutral if, and only if, there exists a minimal cycle $C$ in $H_m$, such that $x \in P_C$.*

*Proof.* In fact, if $x$ is a neutral vertex, then $H_m - x$ is hamiltonian and non-coned. Therefore there exists a minimal cycle $C$ induced in $H_m - x$, and $x \in P_C$. The converse is obvious, from the defintion of poles of $C$. □

**Corollary 2.10.** *Let $H_m$ be a hamiltonian tournament. If $N$ ($Q$, respectively) is the subtournament of the neutral (non-neutral, respectively) vertices of $H_m$, then $N = \cup\,\{P_C \mid C \text{ minimal cycle }\}$ and $Q = \cap\,\{V(C) \mid C \text{ minimal cycle }\}$.*

**Proposition 2.11.** *If $C_1, \ldots, C_r$ are non-coned cycles in a hamiltonian tournament $H_m$, then the subtournament $R = \langle C_1 \cup \ldots \cup C_r \rangle$ spanned by their vertices is hamiltonian.*

*Proof.* If $R = H_m$, then the result is obvious. On the other hand, if $R \neq H_m$, then there exists $x \in H_m - R$. Hence $x$ is a pole of one of the non-coned cycles, and $H_m - x$ is hamiltonian. If $R = H_m - x$, the result follows. If $R \neq H_m - x$, we proceed as before. Since this process has to come to an end, the result is true. □

We now introduce the following definition:

**Definition 2.12.** Let $T_m$ be a tournament of order $m$. If $T_m = Tr_n^*(S^{(1)}, \ldots, S^{(n)})$, with $Tr_n^*$ being the dual transitive tournament of order $n$, and every component $S^{(i)}$ being a singleton or a hamiltonian subtournament, we say we have a *composition in strong components* of $T_m$.

We observe that the tournament $H_m$ is hamiltonian if, and only if, $H_m$ is the only strong component. In the case $Q = \emptyset$, that is, in $H_m$ all the vertices are neutral vertices, then $N = H_m$, hence obviously hamiltonian. On the other hand, if $Q \neq \emptyset$, the situation is not the same as we can see from the next result.

**Theorem 2.13.** *Let $H_m$ be a hamiltonian tournament. If there is at least one non-neutral vertex in $H_m$ (that is, $Q \neq \emptyset$), then the subtournament $N$ of the neutral vertices of $H_m$ is not hamiltonian, or $H_m$ is the composition of two singletons and a hamiltonian component $H'$, with a 3-cycle as quotient.*

*Proof.* See [13]. □

## 3 6-Tournaments having a minimal cycle of length four

To exhibit those tournaments we adopt a constructive viewpoint, starting on a hamiltonian tournament $H$, $|H| = 6$, with minimal cycle $C \subset H$, $|C| = 4$. The first thing to notice is that there is only one hamiltonian structure of order four, which can be characterized by two 3-cycles sharing 2 vertices. Let $C_1 \subset C$ be the 3-cycle having 2 vertices of outdegree 2, and $C_2 \subset C$ be the 3-cycle having two vertices of outdegree 1 (degrees relative to $C$). Thus, in the figure, $C_1 : 1 \to 2 \to 4 \to 1$ and $C_2 : 1 \to 3 \to 4 \to 1$.

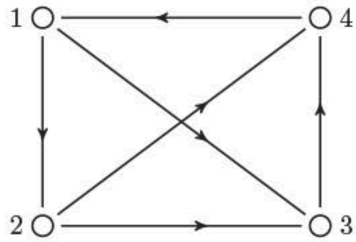


Figure 1: The cycle $C$

There are two vertices in $\langle H - C\rangle$. Since $C$ is minimal each of the cycles $C_j$, $j = 1, 2$, must be coned by at least one of the vertices in $\langle H - C\rangle$. We claim that each $C_j$ is coned by exactly one of those vertices. For if $x \in \langle H - C\rangle$ cones $C_2$, let's say $x \to C_2$, then it must be $x \leftarrow \langle C - C_2\rangle$, otherwise $C$ would be coned by $x$. In particular $x$ does not conne $C_1$, but the remaining vertex in $\langle H - C\rangle$ does. The other adjacency $x \leftarrow C_2$ is treated similarly. We can summarize this by writing $H = \langle\{a_1, a_2\} \cup C\rangle$, and such that $a_j$ cones $C_i$ for $j, i = 1, 2$ if and only if $j = i$.

Analysing the adjacencies between $a_i, C_i$ and $a_1, a_2$ we will show there are precisely five hamiltonian 6-tournaments with a minimal cycle of length 4. We first consider the following subcases:

(1) If $a_2 \to C_2$ it must be $a_2 \leftarrow 2$. The only 3-cycle passing through $a_2$ and some vertices of $C$ is $2 \to a_2 \to 1 \to 2$, which is coned by 3.

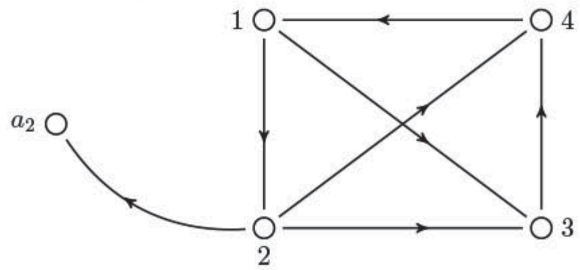


Figure 2: Case $a_2 \to C_2$

(2) If $a_2 \leftarrow C_2$ then $a_2 \to 2$. The 3-cycle $a_2 \to 2 \to 3 \to a_2$ is coned by 1. The 3-cycle $a_2 \to 2 \to 4 \to a_2$ is coned by $a_1$ if and only if

$$\mathrm{ad}(a_1, C_1) = \mathrm{ad}(a_1, a_2). \qquad (2)$$

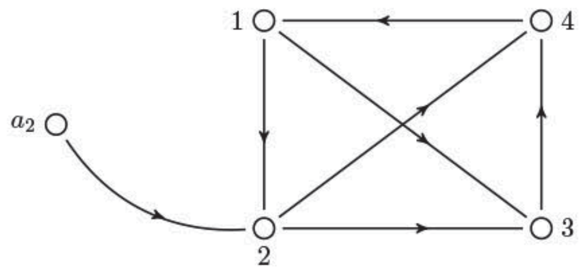


Figure 3: Case $a_2 \leftarrow C_2$

(3) If $a_1 \to C_1$ then $a_1 \leftarrow 3$. The 3-cyle $3 \to a_1 \to 2 \to 3$ is coned by 4. The 3-cycle $3 \to a_1 \to 1 \to 3$ is coned by $a_2$ if and only if

$$\mathrm{ad}(a_2, C_2) = \mathrm{ad}(a_2, a_1). \tag{3}$$

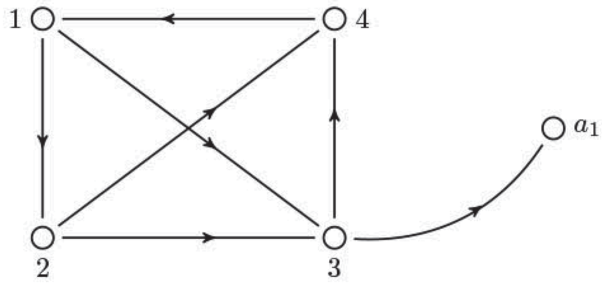


Figure 4: Case $a_1 \to C_1$

(4) If $a_1 \leftarrow C_1$ then $a_1 \to 3$. The 3-cycle $a_1 \to 3 \to 4 \to a_1$ is coned by 2.

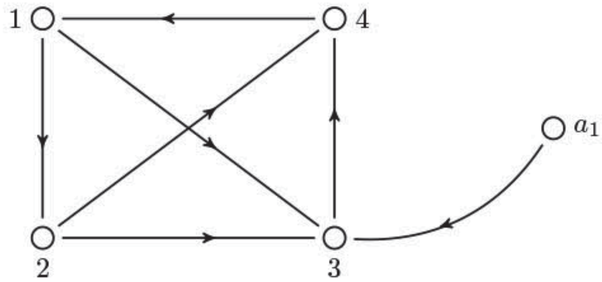


Figure 5: Case $a_1 \leftarrow C_1$

We will combine (1) and (2) with (3) and (4) above. Further we will consider the two possible adjacencies between $a_1$ and $a_2$: $(i)$ for $a_1 \to a_2$ and $(ii)$ for $a_1 \leftarrow a_2$.

(13.i) The 3-cycle $a_1 \to a_2 \to 3 \to a_1$ is coned by 4. Since condition (3) fails the 3-cycle $3 \to a_1 \to 1 \to 3$ is minimal and the cyclic characteristic of this tournament is 3. We call this structure $H^{(13.i)}$, which is characterized by $a_1 \to C_1$, $a_2 \to C_2$ and $a_1 \to a_2$.

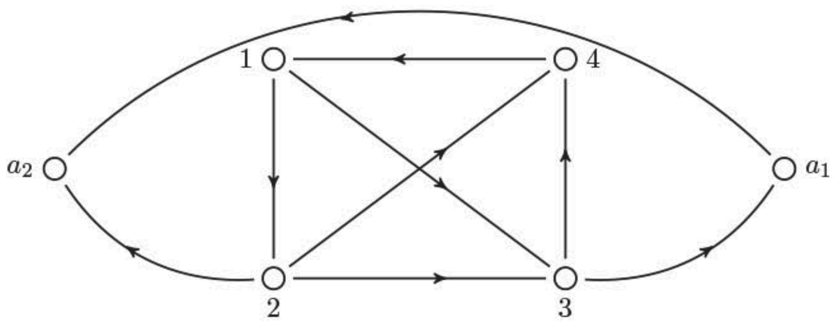


Figure 6: Tournament $H^{(13.i)}$

(13.ii) The only difference from the previous one is that $a_2 \to a_1$. In particular the 3-cycle $3 \to a_1 \to 1 \to 3$ is coned, like all other 3-cycles. Hence this tournament has cyclic characteristic 4. We call it $H^{(13.ii)}$.

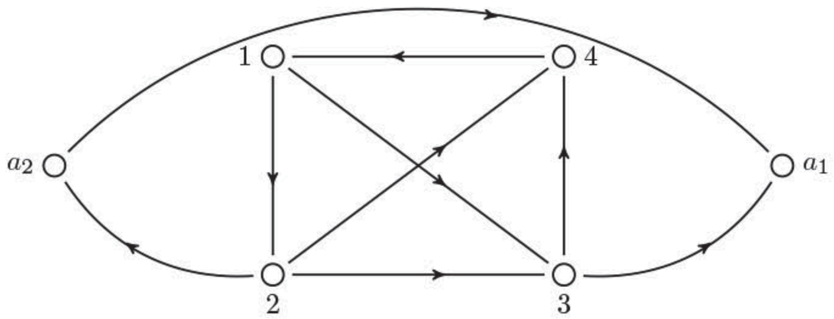


Figure 7: Tournament $H^{(13.ii)}$

(14.i) There are no other 3-cycles besides the ones listed above, and they are all coned. The structure $H^{(14.i)}$ is characterized by $a_1 \leftarrow C_1$, $a_2 \rightarrow C_2$ and $a_1 \rightarrow a_2$. It holds $\mathrm{cc}(H^{(14.i)}) = 4$.

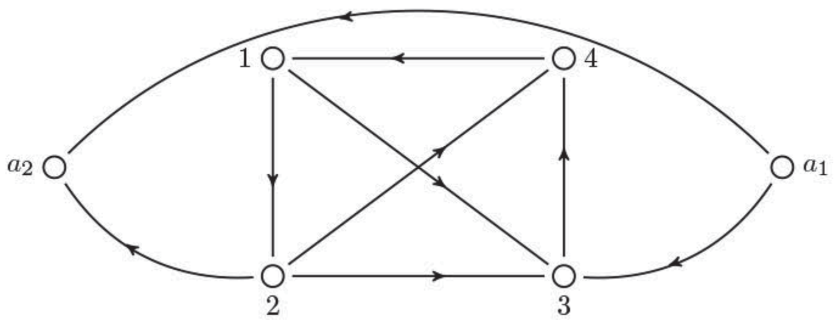


Figure 8: Tournament $H^{(14.i)}$

(14.ii) All 3-cycles are coned. The structure $H^{(14.ii)}$ is characterized by $a_2 \rightarrow C_2$, $a_1 \leftarrow C_1$ and $a_1 \leftarrow a_2$. It holds $\mathrm{cc}(H^{(14.ii)}) = 4$.

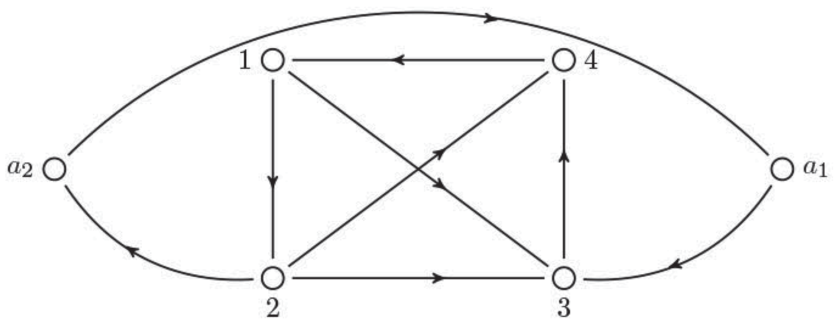


Figure 9: Tournament $H^{(14.ii)}$

(23.i) Condition (2) (which is equivalent to (3) in this case) is satisfyed and all 3-cycles are coned. This tournament $H^{(23.i)}$ is characterized by $a_2 \leftarrow C_2$, $a_1 \rightarrow C_1$ and $a_1 \rightarrow a_2$ and has cyclic characteristic 4.

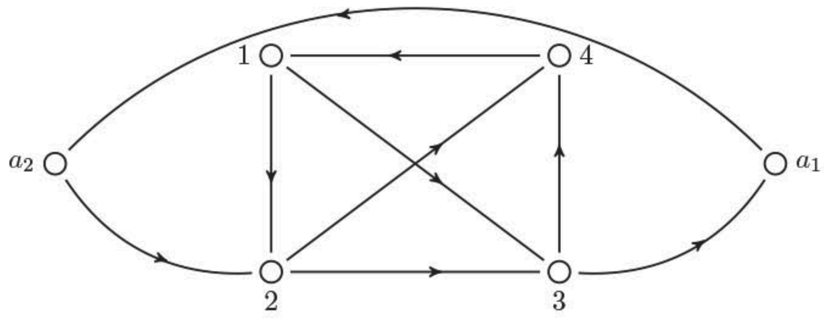


Figure 10: Tournament $H^{(23.i)}$

(23.ii) Conditions (2) and (3) fail, and the 3-cycles $a_2 \to 2 \to 4 \to a_2$ and $a_1 \to 1 \to 3 \to a_1$ are both non-coned. This tournament $H^{(23.ii)}$ has cyclic characteristic 3 and is characterized by $a_2 \leftarrow C_2$, $a_1 \to C_1$ and $a_1 \leftarrow a_2$.

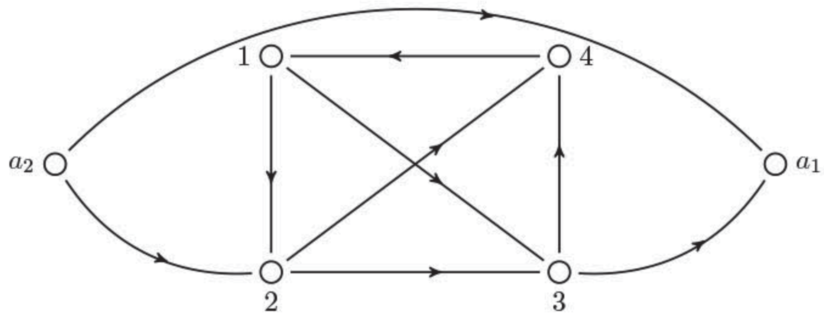


Figure 11: Tournament $H^{(23.ii)}$

(24.i) Condition (2) does not hold and the 3-cycle $a_2 \to 2 \to 4 \to a_2$ is not coned. This tournament $H^{(24.i)}$ is characterized by $a_2 \leftarrow C_2$, $a_1 \leftarrow C_1$, $a_2 \leftarrow a_1$ and has $cc(H^{(24.i)}) = 3$.

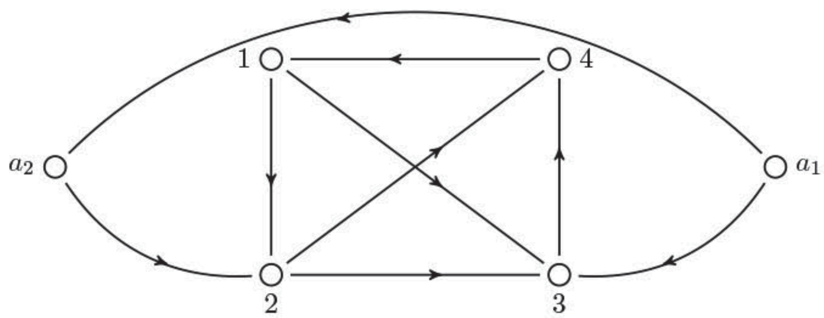


Figure 12: Tournament $H^{(24.i)}$

(24.ii) Condition (2) is satisfied and the resulting structure has cyclic characteristic 4, since $a_2 \to 2 \to 4 \to a_2$ is coned. This tournament $H^{(24.ii)}$ is characterized by $a_2 \leftarrow C_2$, $a_1 \leftarrow C_1$ and $a_2 \to a_1$.

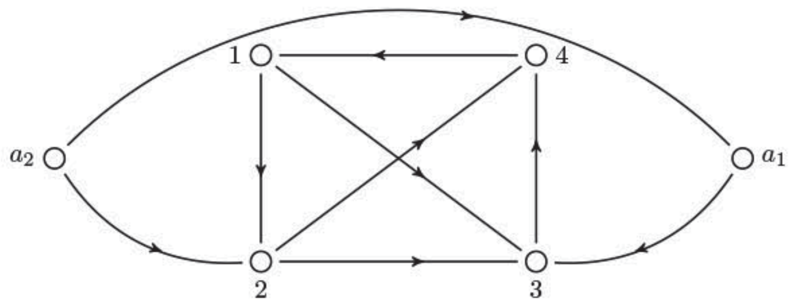


Figure 13: Tournament $H^{(24.ii)}$

So far we have eight structures which contain all possible hamiltonian 6-tournaments having minimal cycles of length 4. We will identify isomorphisms among some of them, so that only five structures will be shown to be non-isomorphic.

**Lemma 3.1.** *Let $\mathcal{T} = \{H^{(13.ii)}, H^{(14.i)}, H^{(23.i)}, H^{(24.ii)}\}$. Then any two tournaments in $\mathcal{T}$ are isomorphic.*

*Proof.* We pick one of these tournaments, say $H^{(13.ii)}$, as reference, and reserve the notation $a_1, a_2$ as well as $C$ for the vertices and minimal 4-cycle used in the contruction of $H^{(13.ii)}$ from scratch. Let $T \in \mathcal{T}$ be arbitrary. We first notice that there is exactly one vertex $a_2' \in T$ having outdegree 4. Second, there are two vertices in $T$ having outdegree 3, but only one of them, which we call $a_1' \in T$, is a successor of $a_2'$. Thus, a prospective isomorphism $\varphi : H^{(13.ii)} \to T$ must send $a_j$ to $a_j'$ for $j = 1, 2$, since outdegrees and adjacency relations are preserved by $\varphi$.

Let $C' = T - \{a_1', a_2'\}$, and assume for the moment $C'$ is a cycle (of length 4). There is only one way to define $\varphi$ on the vertices of $C$ and such that $C$ is sent onto $C'$ isomorphically. Hence $\varphi$ is a bijection between the sets of vertices of $H^{(13.ii)}$ and $T$, and a morphism between the subtournaments $\langle a_1, a_2 \rangle \simeq \langle a_1', a_2' \rangle$ and $C \simeq C'$. Now we verify that adjacencies of a vertex in $\{a_1, a_2\}$ and a vertex in $C$ are preserved by $\varphi$. Let $C' = C_2' \cup C_1'$ be the union of two 3-cycles, and $C_2'$ be the one having two vertices of outdegree 1 relative to $C'$. Since the 4-cycle $C'$ is non-coned, and hence minimal for $\mathrm{cc}(T) = 4$, both of $a_1', a_2'$ must conne one of $C_1', C_2'$, and an argument used earlier gives us that $a_i' \to C_j'$ for some indices $i, j \in \{1, 2\}$ (the proposed adjacency comes from the outer degrees of $a_1', a_2'$, which are greater than or equal to 3).

Suppose by absurd that it holds $a_2' \to C_1'$. Then it also hold $a_1' \to C_2'$ and $a_2' \leftarrow 3'$, where $3' = \varphi(3)$. But since $3'$ is the predecessor of $a_2'$ its outdegree is 3, so $3' \to a_1'$. There lies the contradiction for $3' \in C_2'$, thus $3' \leftarrow a_1'$. The right adjacencies are then $a_2' \to C_2'$ and $a_1' \to C_1'$. These are the same adjacencies brought by $\varphi$ from $H^{(13.ii)}$. We conclude that $H^{(13.ii)} \simeq T$ as long as $C'$ is a 4-cycle.

Now we exhibit this 4-cycle in each case: for $T = H^{(14.i)}$, $C'$ is $a_1 \to a_2 \to 3 \to 4 \to a_1$; for $T = H^{(23.i)}$, $C'$ is $1 \to a_2 \to 2 \to 4 \to 1$; and for $T = H^{(24.ii)}$, $C'$ is $3 \to a_2 \to 2 \to a_1 \to 3$. This finishes with the proof. □

**Lemma 3.2.** *The collection $\mathcal{U} = \{H^{(13.i)}, H^{(13.ii)}, H^{(14.ii)}, H^{(23.ii)}, H^{(24.i)}\}$ is constituted of 5 non-isomophic hamiltonian tournaments of order 6. An arbitrary hamiltonian tournament of order 6 has a minimal 4-cycle if and only if it is isomorphic to one of the tournaments in $\mathcal{U}$.*

*Proof.* Among the tournaments in $\mathcal{U}$ only $H^{(13.ii)}$ and $H^{(14.ii)}$ have cyclic characteristic 4. Since the former has 6 3-cycles and the latter has 4 3-cycles we see that neither of them is isomorphic to any of the other tournaments in $\mathcal{U}$.

The tournament $H^{(23.ii)}$ has eight 3-cycles, namely $C_1$, $C_2$, $a_2 \to 2 \to 3 \to a_2$, $3 \to a_1 \to 2 \to 3$, $a_2 \to a_1 \to 4 \to a_2$, $a_2 \to a_1 \to 1 \to a_2$, $a_2 \to 2 \to 4 \to a_2$ and $3 \to a_1 \to 1 \to 3$, the last two of them being non-coned. Since $H^{(13.i)}$ and $H^{(24.i)}$ both have six 3-cycles each one we conclude that $H^{(23.ii)}$ is non-isomoprhic to the other structures in $\mathcal{U}$.

We now get to the distinction between $H^{(13.i)}$ and $H^{(24.i)}$. Besides the common number of 3-cycles, both of them enjoy the same number of 4-cycles (6). The number of non-coned 3-cycles, and 4-cyles, is the same in each case. The outdegrees of the vertices follow the pattern 4,3,3,2,2,1. Thus an effort to show these tournaments are non-isomorphic seem to get down to a direct construction of a tentative isomoprhism between them. Despite this approach is not hard in this case, we devise a mechanism that might be useful in other similar contructions in generic order. The idea is to use the already collected information on 3-cycles and 4-cycles.

For a given hamiltonian tournament $H$ let $G = G(H)$ be the directed graph whose vertices are the 3-cycles of $H$. If $\alpha$ and $\beta$ are vertices of $G$, set $\alpha \to \beta$ if and only if $\langle \alpha \cup \beta \rangle$ is a 4-cycle of $H$ and such that $\alpha$ plays the role of $C_1 \subset C$. Notice that $\alpha \to \beta$ is equivalent to $\alpha$ and $\beta$ share a common edge in $H$ and the other vertex of $\alpha$ (outside that edge) preccedes the other vertex of $\beta$. It is not hard to see the structures of $G^\sigma = G(H^\sigma)$ in each case of $\sigma \in \{(13.i), (24.i)\}$:

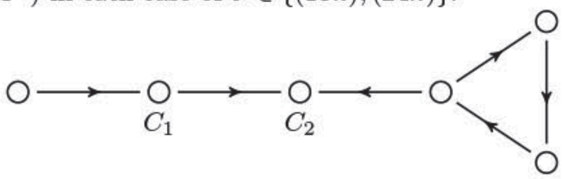


Figure 14: Graph $G^{(13.i)}$

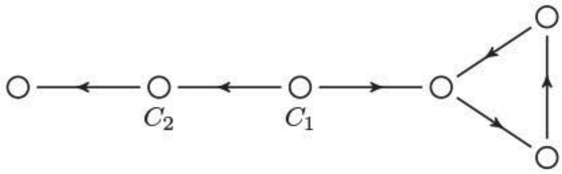


Figure 15: Graph $G^{(24.i)}$

The comparison of the above figures shows us that these two graphs, hence their associated hamiltonian tournaments, are not isomorphic. This finishes with the Lemma's proof. □

In [14] J. W. Moon presents a list of drawing that illustrates the nonisomorphic tournaments $T_n^{(r)}$ $(n \leq 6)$, with their score vectors, the number of ways of labeling their vertices, and their automorphism groups (recall that in there, not all the arcs are included in the drawing; if an arc joining two vertices has not been drawn, then it is to be understood that the arc is oriented from the higher vertex to the lower vertex). If the tournament is hamiltonian it is denoted by $H_n^{(r)}$, instead of $T_n^{(r)}$. Comparing his findings with our conctruction we can state the Lemma:

**Lemma 3.3.** *Keep the notation in* [14]. *Then* $H^{(13.i)} \simeq H_6^{(33)}$, $H^{(13.ii)} \simeq H_6^{(41)}$, $H^{(14.ii)} \simeq H_6^{(21)}$, $H^{(23.ii)} \simeq H_6^{(56)}$ *and* $H^{(24.i)} \simeq H_6^{(35)}$.

Following the idea presented in the proof of Lemma 3.2, we denote by $G^\sigma = G(H^\sigma)$, with $\sigma \in \{(13.i), (13.ii), (14.ii), (23.ii), (24.i)\}$, the associated graph of 3-cycles of each of the hamiltonian 6-tournaments treated in the Lemma. Figures 16, 17 and 18 add up to figures 14, 15, exhibiting the structure of the graphs $G^\sigma$.

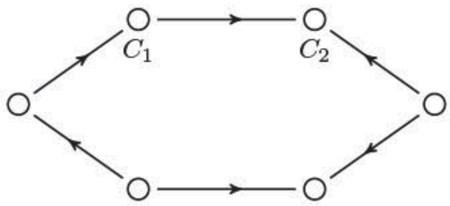


Figure 16: Graph $G^{(13.ii)}$

Figure 17: Graph $G^{(14.ii)}$

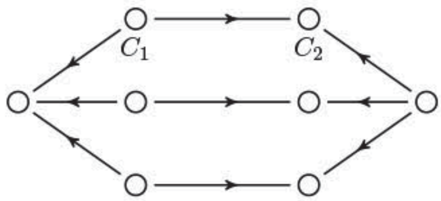


Figure 18: Graph $G^{(23.ii)}$

**Remark 3.4.** Much of the information surveyed on the hamiltonian tournaments is readily available from the graphs on figures 14 through 18. For instance, the number of 3-cycles and 4-cycles in $H^\sigma$ is obviously the number of vertices and edges, respectively, of $G^\sigma$. Further, the number of vertices of $H^\sigma$ that do not conne a given 3-cycle $D \in G^\sigma$ is the degree of $D$ as a vertex of $G^\sigma$. In particular, $H^{(23.ii)}$ has two non-coned 3-cycles while $H^{(13.i)}$ and $H^{(24.i)}$ have each of them one non-coned 3-cycle.

The issue of an edge $\beta$ of $G^\sigma$ (i.e., a 4-cycle on $H^\sigma$) joining the vertices $C$ and $D$ (both of which are 3-cycles on $H^\sigma$) being coned is not so immediate. If one of the vertices $C$ or $D$ has outdegree 1 and the other has outdegree 2 then $\beta$ is coned as a 4-cycle, for there exists one vertex $p \in H^\sigma$ that cones both of $C$ and $D$, and hence $\beta$. This is the case of the external edges in $G^{(14.ii)}$ (Figure 17) and the first edge on the left on $G^{(13.i)}$ and $G^{(24.i)}$ (Figures 14 and 15). Similarly, if one of $C$ or $D$ has outdegree 3 then this 3-cycle is not coned, and so is $\beta$ (though $\beta$ is not minimal in that case). Such 4-cycles happen in $G^{(23.ii)}$, $G^{(13.i)}$ and $G^{(24.i)}$. Though if the vertices $C$ and $D$ both have outdegree 2 it seems unlikely one can tell whether $\beta$ is coned or not without looking at the inner structure of $H^\sigma$.

Recall that we denote by $N = N(H)$ ($Q = Q(H)$) the subtournament of a hamiltonian tournament $H$ constituted by the neutral (non-neutral) vertices of $H$. The next table is obtained by a simple inspection on the structure of these tournaments.

| $H$ | $V(N(H))$ | $V(Q(H))$ |
|---|---|---|
| $H^{(13.i)}$ | $\{a_1, a_2, 2, 4\}$ | $\{1, 3\}$ |
| $H^{(13.ii)}$ | $\{a_1, a_2, 3, 4\}$ | $\{1, 2\}$ |
| $H^{(14.ii)}$ | $\{a_1, a_2\}$ | $\{1, 2, 3, 4\}$ |
| $H^{(23.ii)}$ | $\{a_1, a_2, 1, 2, 3, 4\}$ | $\emptyset$ |
| $H^{(24.i)}$ | $\{a_1, a_2, 1, 3\}$ | $\{2, 4\}$ |

Table 1: Neutral and non-neutral vertices

**Corollary 3.5.** *We have: (i)* $H_6^{(21)} = A_6$ *and* $Q = A_4$*; (ii)* $H_6^{(33)}$ *is a simple tournament and* $N \simeq Tr_4$*; (iii)* $H_6^{(35)}$ *is a simple tournament and* $N = Tr_4$*; (iv)* $H_6^{(41)}$ *is a simple tournament and* $N \simeq Tr_4$*;*

Summing up, we have the following theorem.

**Theorem 3.6.** *Let* $H_6$ *be a hamiltonian tournament of order* 6*, having minimal cycles of length four. Then:*
*(1)* $H_6$ *is unique if it has the maximal number of non-neutral vertices. Moreover, in this case,* $H_6$ *is the bineutral tournament* $A_6 \simeq H_6^{(21)}$*;*
*(2)* $H_6$ *is unique if it has four minimal cycles of length* 4*. This tournament has no minimal cycle of length 3, and does not have the maximal number of non-neutral vertices. In this case* $H_6$ *has exactly two non-neutral vertices and it holds* $H_6 \simeq H_6^{(41)}$*.*
*(3)* $H_6$ *is unique if it has three minimal cycles of length* 4*. This tournament also admits minimal*

*cycles of length* 3*, and all its vertices are neutral. It holds* $H_6 \simeq H_6^{(56)}$.
(4) *There are two* 6*-tournaments having just one minimal cycle of length* 4 *and admiting minimal cycles of length* 3*. Both of them have two non-neutral vertices and they are isomorphic to* $H_6^{(33)}$ *and* $H_6^{(35)}$.

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