

EITHER DIGRAPHS OR PRE-TOPOLOGICAL SPACES?

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RESUMO: Neste artigo mostramos que os digrafos podem ser identificados, de um modo natural, com espaços pre-topológicos finitos. Mostramos também que com esta identificação a Teoria de Homotopia Regular é a mais apropriada a ser usada quando se trabalha com digrafos. Em particular obtemos caracterizações gráficas e estruturais para algumas classes de torneios, mostrando a importância desta nova abordagem.

Palavras-chave: Digrafos, Espaços Pré-topológicos, Homotopia Regular para Digrafos, Torneios.

ABSTRACT: In this paper we show that digraphs can be identified, in a natural way, to finite pre-topological spaces. We also show that with this identification the Regular Homotopy Theory is the most appropriate one to be used when dealing with digraphs. We give some combinatorial applications of the homotopy theory of pre-topological spaces to digraphs. In particular we get structural and graphical characterizations for some classes of tournaments, showing the importance of this new approach.

Keywords: Digraphs, Pre-topological spaces, Regular Homotopy for Digraphs, Tournaments.

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1. INTRODUCTION

In Graph Theory sometimes one has to introduce certain structures in order to study or to obtain successful applications. In [24], for instance, tournaments are studied using an algebraic approach so that they are considered as algebras of a special kind.

Sometimes a topological approach is used. Since graphs can always be realized in the euclidean space, if one is interested in studying them from a homotopical viewpoint, the usual procedure is to consider them just as 1-complexes. Demaria and his collaborators used a different approach, taking in account just the combinatorial data furnished by the graphs, obtaining what they have called the *Regular Homotopy Theory for Digraphs*.

It is known that one can introduce structures in a set which are weaker than a topology. For example, we can consider certain sets as *pre-topological spaces*. Considering the pre-continuous maps we can construct the corresponding homotopy theory. In this paper we show that, in a natural way, to any digraph (directed graph) D we can associate two pre-topological spaces $P(D)$ and $P^*(D)$. In fact we will show that the class of the digraphs can actually be identified with the class of the finite pre-topological spaces. As a matter of fact, the main purpose of this paper is to stablish the point that *a finite pre-topological space is just a digraph, and vice versa*.

Since we have this identification, it seems proper to use the corresponding homotopy theory when a digraph is considered as a pre-topological space in order to study them from a homotopical point of view. In this paper we will show that this homotopy theory is exactly that introduced by Demaria. Moreover we shall present several applications, showing this new approach is relevant and effective leading to some important new results.

In section 2, we give the basic definitions, the notation and we prove the equivalence between the class of the digraphs and the class of the finite pre-topological spaces.

In section 3, we present a summary of the Regular Homotopy for Digraphs, which was introduced by D. C. Demaria and his collaborators, showing that it coincides with the homotopy theory for pre-topological spaces, since the o -regular maps are just the pre-continuous maps.

In the last four sections we present several recent results, showing how this identification of digraphs with finite prespaces and the proper use of the corresponding homotopy theory has led to important theorems characterizing certain families of tournaments and digraphs, as

well as many positive results on the reconstruction problem for tournaments.

2. PRE-TOPOLOGICAL SPACES AND DIGRAPHS

We shall recall some basic definitions and properties of pre-topological spaces. The terminology is fairly standard following [12,24].

Definition 2.1. Let S be a non-empty set. For every $x \in S$, let F_x be a filter on S such that $\overline{\{x\}} \leq F_x$, where $\overline{\{x\}}$ is the filter generated by $\{x\}$. We call the filter F_x the *neighborhood filter* of the point x . The collection $\mathcal{P} = \{F_x \mid x \in S\}$ is said to be a *pre-topology* on S and the pair $P = (S, \mathcal{P})$ a *pre-topological space* or *prespace*. The prespace P is called *principally filtered* or *p.f. prespace* if each filter $F_x \in \mathcal{P}$ is principal.

It is obvious that any topological space is also a prespace.

Remark 2.1. If for every $A \subseteq S$ we put $\text{cl}(A) = \{x \in S \mid F_x \wedge \overline{A} \neq \emptyset\}$, then we have defined a *closure operator* on S . In this way a prespace becomes a *Čech closure space* (see [12,13]). The converse is also true.

To introduce pre-continuous maps between prespaces we follow the same process as to define continuous maps between spaces.

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Definition 2.2. Let $P = (S, \mathcal{P})$ and $P' = (S', \mathcal{P}')$ be two given prespaces. A map $f : S \rightarrow S'$ is said to be *pre-continuous* if, for every $x \in S$, we have $f(F_x) \leq F'_{f(x)}$, where $f(F_x)$ is the f -image of the filter F_x .

Remark 2.2. If we consider P and P' as closure spaces then we have the following result:

$f : S \rightarrow S'$ is precontinuous if and only if
 $(\forall A' \subseteq S') (\forall B' \subseteq S') A' \cap \text{cl}(B') = \emptyset \Rightarrow f^{-1}(A') \cap \text{cl}(f^{-1}(B')) = \emptyset$.

The next definition shows how one can associate, in a natural way, to any given digraph D two prespaces $P(D)$ and $P^*(D)$.

Definition 2.3. Let S be a finite non-empty set and \mathcal{A} a set of ordered pairs $(x, y) \in S \times S$, such that $x \neq y$. We say the pair $D = (S, \mathcal{A})$ is a *directed graph* or *digraph*. The elements of S are the *vertices* of D , the cardinality of S the *order* of D and the elements of \mathcal{A} the *arcs* of

D . Moreover we write $x \rightarrow y$ instead of (x, y) and in this case we call x a *predecessor* of y and y a *successor* of x . By D^* we shall denote the dual digraph $D^* = (S, \mathcal{A}^*)$, where $(x, y) \in \mathcal{A}^*$ if and only if $(y, x) \in \mathcal{A}$.

Definition 2.4. Let $D = (S, \mathcal{A})$ and $D' = (S', \mathcal{A}')$ be two digraphs. A map $f : S \rightarrow S'$ is said to be a *homomorphism* between the digraphs D and D' if, for every $x, y \in S$, $x \rightarrow y$ we have either $f(x) \rightarrow f(y)$ or $f(x) = f(y)$.

Definition 2.5. Let $D = (S, \mathcal{A})$ be a digraph. We call *prespaces associated to D* the two prespaces $P(D) = (S, \mathcal{P})$ and $P^*(D) = (S, \mathcal{P}^*)$, where the pretopologies $\mathcal{P} = \{F_x\}$ ($x \in S$) and $\mathcal{P}^* = \{F_x^*\}$ ($x \in S$) are defined in the following way, respectively:

For every vertex $x \in S$, we consider the neighborhoods:

$$A(x) = \{x\} \cup \{y \in S \mid x \rightarrow y\} \text{ and } A^*(x) = \{x\} \cup \{y \in S \mid y \rightarrow x\},$$

then we put $F_x = \overline{A(x)}$ and $F_x^* = \overline{A^*(x)}$.

Thus given a digraph D we can, in a natural way, introduce a structure of prespace. We now will describe how a finite prespace can be considered as a digraph, also in a natural way.

The next definition shows how one can associate, in a natural way, to any given finite prespace P two digraphs $D(P)$ and $D^*(P)$.

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Definition 2.6. Let $P = (S, \mathcal{P})$ be a finite prespace. One can in a natural way associate to P two digraphs $D(P) = (S, \mathcal{A})$ and $D^*(P) = (S, \mathcal{A}^*)$, which are dually directed. In fact every filter F_x ($x \in S$) is a principal filter $\overline{A_x}$ of base A_x and, for $x \in S, y \in S, x \neq y, y \in A_x$, we put $x \rightarrow y$ or $y \rightarrow x$, respectively. The digraphs so obtained are called the *digraphs associated to P* .

In particular we have $P(D) = P^*(D)$ if and only if $D = D^*$: in this case D may be considered an undirected graph.

Definition 2.7. We observe that any homomorphism between two digraphs D and D' is a precontinuous map between the prespaces $P(D)$ and $P(D')$ (or $(P^*(D)$ and $P^*(D')$), and vice versa.

Theorem 2.1. *The class of all digraphs can be identified, in a natural way, with the class of all finite prespaces.*

Proof. The result follows immediately from the last two definitions. In fact, if we start with a given digraph D , then take its associated prespace $P(D)$ and then take the associated digraph $D(P(D))$ we recuperate again the digraph D . On the other hand, starting with a given prespace P , then taking its associated digraph $D(P)$ and then the associated prespace $P(D(P))$ we get again the prespace P . \square

3. REGULAR HOMOTOPY THEORY FOR DIGRAPHS

Demaria and his collaborators, in order to introduce the *Regular Homotopy Theory for Digraphs*, they had to define the concept of o -regular (and o^* -regular) maps. We shall describe here an outline of their theory (see [2, 3, 4, 5]).

Definition 3.1. A function $f : P \rightarrow D$ between a prespace P and a digraph D is o -regular (o^* -regular) if and only if, for each pair of different vertices x and y of D such that $x \not\rightarrow y$, we have: $f^{-1}(x) \cap \text{cl}(f^{-1}(y)) = \emptyset$ ($\text{cl}(f^{-1}(x)) \cap f^{-1}(y) = \emptyset$).

Definition 3.2. We say two o -regular (o^* -regular) functions f and f' are o -homotopic (o^* -homotopic) if there exists a homotopy from f to f' which is an o -regular (o^* -regular) function.

Definition 3.3. In definition 3.1, if we take P to be I^n , the n -cube and we consider the homotopic classes of n -loops on (D, x) which are o -regular (o^* -regular), then we obtain the homotopy groups which are called the o -regular (o^* -regular) homotopy n -groups of the digraph D at the vertex-base x . These groups are denoted by $Q_n(D, x)$ ($Q_n^*(D, x)$).

Remark 3.1. The homotopy groups $Q_n(D, x)$ and $Q_n^*(D, x)$ of a weakly connected digraph D do not depend on the vertex-base x . So we denote them by $Q_n(D)$ and $Q_n^*(D)$.

As we have seen in the previous section, digraphs are actually finite prespaces. So the natural homotopy theory to be done on a digraph should be the one that comes naturally from this identification.

As a matter of fact, one can develop a homotopy theory for prespaces which is similar to the classical one for topological spaces. To this purpose we call n -paths of the prespace P the pre-continuous maps $f : I^n \rightarrow P$. For the rest we have only to replace the terms “topological space” and “continuous map” respectively by “prespace” and “pre-continuous map”. Therefore, given a prespace P and a point $x \in P$,

we can construct the *homotopy groups* $\pi_n(P, x)$ of P at the point-base x .

The next proposition show that this homotopy theory and that obtained by Demaria are exactly the same.

Proposition 3.1. *Let P be a prespace and D a digraph. Let $P(D)$ be the prespace associated to D . A function $f : P \rightarrow D$ is o -regular (resp. o^* -regular) if and only if f is a precontinuous map between the prespaces P and $P(D)$ (resp. P and $P^*(D)$).*

Proof. The result follows from the definition of the pre-topological structure on $P(D)$ (resp. $P^*(D)$) and the the fact that a map $f : P \rightarrow D$ between a prespace P and a digraph D is o -regular (o^* -regular) if and only if, for each pair of different vertices x and y of D such that $x \not\rightarrow y$, we have: $f^{-1}(x) \cap \text{cl}(f^{-1}(y)) = \emptyset$ ($\text{cl}(f^{-1}(x)) \cap f^{-1}(y) = \emptyset$) \square

Remark 3.2. This result is in fact saying that the homotopy groups $\pi_n(P(D), x)$ ($\pi_n(P^*(D), x)$) are isomorphic to the o -regular (o^* -regular) homotopy groups of the digraph D at the vertex-base x , $Q_n(D, x)$ ($Q_n^*(D, x)$).

Given D a weakly connected digraph, the regular homotopy groups $Q_n(D)$ and $Q_n^*(D)$ can be calculated using the classical homotopy groups $\pi_n(|K_D|)$ of a suitable polyhedron K_D , since the following theorems hold:

Theorem 3.1. *The homotopy groups $Q_n(D)$ and $Q_n^*(D)$ are isomorphic (see [5]).*

Theorem 3.2. *$Q_n(D)$ is isomorphic to the classical homotopy group $\pi_n(|K_D|)$ of the polyhedron of a “suitable” simplicial complex K_D associated with D (see [6]).*

We shall describe shortly how to get this suitable simplicial complex K_D .

If $H \subset D$ we say H is *headed* if there exists a vertex v in H such that $v \rightarrow H \setminus v$. And H is said to be *totally headed* if for every $A \subset H$ with $A \neq \emptyset$, we have that A is headed. The simplexes in K_D are the ones generated by the totally headed subdigraphs of D .

The proof of these two theorems is hard and very long (see [2, 3, 4, 5, 6]). We just say that the first step is to prove some propositions (*the normalization theorems*), which are similar to the simplicial approximation theorems for continuous maps between polyhedra, which allow to choose special representatives in each regular homotopy class. Finally, using them we the desired isomorphism is obtained.

We give some examples of digraphs and their corresponding associated polyhedra.

1. If D is the undirected graph of the edges of a triangle, $|K_D|$ is homeomorphic to E^2 (the full disk) and $Q_n(D) = 0$ ($n \geq 1$).
2. If D is the undirected graph of the edges of a square, $|K_D|$ is homeomorphic to S^1 (the circle) and $Q_n(D) \simeq \pi_n(S^1)$.
3. If D is the digraph with vertices a, b, c and arcs $a \rightarrow b, b \rightarrow c, c \rightarrow a$, $|K_D|$ is also homeomorphic to S^1 .
4. If D is the undirected graph of the edges of a tetrahedron, $|K_D|$ is homeomorphic to E^3 and $Q_n(D) = 0$ ($n \geq 1$).
5. If D is the undirected graph of the edges of an octahedron, $|K_D|$ is homeomorphic to S^2 (the sphere) and $Q_n(D) \simeq \pi_n(S^2)$.
6. If D is the digraph with vertices a, b, c, d and arcs $a \rightarrow b, b \rightarrow c, c \rightarrow d, d \rightarrow a, b \rightarrow d, d \rightarrow b, a \rightarrow c, c \rightarrow a$, $|K_D|$ is also homeomorphic to S^2 .

4. TOURNAMENTS

In this section we recall some results and definitions for the special case in which the digraph D is a tournament.

Definition 4.1. A digraph D is called a *tournament* if every pair of different vertices of D is joined by one and only one arc. A tournament T is called *hamiltonian* if it contains a spanning cycle, i.e. a cycle through all the vertices of T .

Definition 4.2. A tournament T is called *regular* if, for each vertex $x \in T$, the number of the predecessors and successors of x is the same (hence the order of T is odd). A tournament T is called *highly regular* if there exists a cyclical ordering $x_1, x_2, \dots, x_{2m+1}, x_1$ on the vertices of T such that $x_i \rightarrow x_j$ if and only if x_j is one of the first m successors of x_i in the cyclical ordering of T .

Every tournament can be endowed with an algebraic structure, in a natural way (see [28]). In fact:

Proposition 4.1. A tournament T becomes the commutative groupoid $A(T)$ (see [7]) if we define the following binary operation \circ :

$$(\forall x, y \in T) x \circ y = y \circ x = \begin{cases} x, & \text{if } x \rightarrow y \text{ or } x = y; \\ y, & \text{if } y \rightarrow x. \end{cases}$$

Proof. See [28]. □

Remark 4.1. Similarly we can associate with T the dual commutative groupoid $A^*(T)$, defining:

$$(\forall x, y \in T) x \circ y = y \circ x = \begin{cases} x, & \text{if } x \rightarrow y \text{ or } x = y; \\ y, & \text{if } y \rightarrow x. \end{cases}$$

Remark 4.2. Every homomorphism between two tournaments T and T' is also an algebraic homomorphism between the commutative groupoids $A(T)$ and $A(T')$ ($A^*(T)$ and $A^*(T')$), and vice versa.

Let T_n and T'_m be two tournaments of order n and m , respectively, and $p : T_n \rightarrow T'_m$ a surjective homomorphism. We have that the groupoid $A(T'_m)$ is isomorphic to the quotient $A(T_n)/p$. Therefore, considering the m preimages of the vertices y_1, y_2, \dots, y_m of T'_m and putting $S^{(i)} = p^{-1}(y_i)$ for each $i = 1, 2, \dots, m$, we can partition the n vertices of T_n into m disjoint subtournaments $S^{(1)}, S^{(2)}, \dots, S^{(m)}$ of equivalent vertices. Moreover, if $y_i \rightarrow y_j$ we get $S^{(i)} \rightarrow S^{(j)}$, where $S^{(i)} \rightarrow S^{(j)}$ means that the vertices of $S^{(i)}$ are all predecessors of all the vertices of $S^{(j)}$.

Definition 4.3. Under the same assumptions, we write $T_n = T'_m(S^{(1)}, S^{(2)}, \dots, S^{(m)})$ and call T_n the *composition* of the m tournaments $S^{(1)}, S^{(2)}, \dots, S^{(m)}$ with the tournament T'_m . The subtournaments $S^{(1)}, S^{(2)}, \dots, S^{(m)}$ are the *components* of T_n and the tournament T'_m is called a *quotient* of T_n .

A tournament T_n is said to be *simple* if the composition $T_n = T'_m(S^{(1)}, S^{(2)}, \dots, S^{(m)})$ implies either $m = 1$ or $m = n$.

The following properties hold.

Proposition 4.2. *For every non-trivial tournament T there is precisely one non-trivial simple quotient tournament T^* , called the simple quotient tournament related to T .*

Remark 4.3. For $h \geq 3$, where h is the order of T^* , there is precisely one partition of the vertices of T into h components. Whereas, for $h = 2$ the partition is not unique.

Remark 4.4. Let T be a tournament and T' a quotient tournament of T . Then there exists a subtournament T^* of T isomorphic to T' .

Proposition 4.3. *A tournament is hamiltonian if and only if every one of its non-trivial quotient tournaments is hamiltonian.*

5. SIMPLY DISCONNECTED TOURNAMENTS

Now we present the characterization given by Burzio and Demaria (see [7, 8]) for the class of tournaments, whose fundamental group is non-trivial.

Definition 5.1. A tournament T is called *simply connected* (*simply disconnected*) if its first homotopy group $Q_1(T)$ is trivial (non-trivial).

Remark 5.1. Since T is weakly connected, the fundamental group is independent of the choice of the vertex base, allowing us to use the notation $Q_1(T)$.

To obtain $Q_1(T)$ we note that $Q_1(T)$ is isomorphic to $\pi_1(|K_T|)$, where K_T is the complex whose vertex set is T and whose simplexes are spanned by the transitive subtournaments of T . Moreover $\pi_1(|K_T|)$ can be calculated by using *edge-loops* made up of edges of K_T .

In detail, we must remember that an edge-loop, based at a vertex x of K_T , is a sequence $xx^1x^2 \dots x^kx$ of vertices of K_T , in which each consecutive pair $x^i x^{i+1}$ spans a simplex of K_T .

Two edge-loops, based at x , are *homotopic* if we can obtain one from the other by a finite number of the following operations:

1. a repeated vertex yy can be changed to y and vice versa;
2. if three consecutive vertices wyz span a simplex of K_T , they may be replaced by the pair wz and vice versa.

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The set of equivalent classes of edge-loops, based at x , becomes a group under the multiplication:

$$\{xx^1x^2 \dots x^kx\}\{xy^1y^2 \dots x^h x\} = \{xx^1x^2 \dots x^k xxy^1y^2 \dots y^h x\},$$

which is isomorphic to $\pi_1(|K_T|)$.

In this case, since T is a tournament, any pair of different vertices of T generates a 1-simplex of K_T and any transitive subtournament of order 3 of T a 2-simplex of K_T .

A structural characterization for the simply disconnected tournaments is obtained in the following theorem (see [8]).

Theorem 5.1. *A tournament T is simply disconnected if and only if its simple quotient tournament is highly regular.*

Proof. The proof of this theorem is obtained in three steps.

S₁. A tournament T is simply disconnected if and only if every one of its non-trivial quotients T' is simply connected.

In fact, by identifying T' as a subtournament T'' of T and by choosing a vertex $x \in T''$ as the vertex-base, each edge loop of $|K_T|$ is homotopic to its projection on $|K_{T''}| \subseteq |K_T|$.

S₂. A non-trivial highly regular tournament T_{2m+1} is simply disconnected.

For $m = 1$, T_3 is the 3-cycle and $|K_{T_3}| = S^1$. So $Q_1(T_3) = \pi_1(S^1) \sim \mathbb{Z}$.

We show that $|K_{T_{2m-1}}|$ is a deformation retract of $|K_{T_{2m+1}}|$, by using the following topological tools.

Lemma 5.1. *If X is a deformation retract of a topological space S and Y a deformation retract of X , then Y is a deformation retract of S .*

Lemma 5.2. *Let S be a topological space, X and Y two closed subspaces of S , such that $X \cup Y = S$. If Z is a deformation retract of Y , such that $X \cap Y \subseteq Z$, then $X \cup Z$ is a deformation retract of S .*

Hence, by induction on m , we prove that the polyhedron $|K_{T_3}| = S^1$ is a deformation retract of $|K_{T_{2m+1}}|$. So we have $Q_1(T_{2m+1}) \sim \mathbb{Z}$.

S₃. The simple quotient tournament related to a simply disconnected tournament is highly regular.

This proposition is proved by using induction on the order n ($n \geq 3$) of the tournament T .

For $n = 3$, let T_3 be simply disconnected. Then T_3 is the 3-cycle and it is also highly regular. Therefore it coincides with its simple quotient.

Assume that, for each simply disconnected tournament T_n , the simple quotient tournament T' related to T_n is highly regular. Consider a simply disconnected tournament T_{n+1} . Then for a suitable ordering x_1, x_2, \dots, x_{n+1} of the vertices of T_{n+1} the edge-loop $x_1x_2x_3x_1$ is not nullhomotopic in $|K_{T_{n+1}}|$. The same edge-loop is not nullhomotopic in

$|K_{T_n}|$, where $T_n = T_{n+1} - x_{n+1}$.

By the inductive hypothesis, the simple quotient tournament T' related to T_n is highly regular.

Finally, by considering the orientation of the arcs between x_{n+1} and any vertex $x_i \in T_n$, it results that the simple quotient tournament of T_{n+1} is also highly regular.

From steps 1,2 and 3 it follows the theorem. □

As a consequence we have the following:

Proposition 5.1. *A simply disconnected tournament is hamiltonian.*

In [7] Burzio and Demaria also gave a graphical characterization for the simply disconnected tournaments. At first, we observe that a cycle whose edge-loop becomes the base of a cone in the associate polyhedron is clearly nullhomotopic. In particular a cycle, whose vertices are included in a component of the tournament, is nullhomotopic. So we put the following definitions:

Definition 5.2. Let C be a cycle of a tournament T . C is said to be *coned* by a vertex x if there exist a vertex $x \in T - C$ such that either $x \rightarrow C$ or $C \rightarrow x$. Otherwise C is said to be *non-coned*.

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In this way Burzio and Demaria characterized the simply disconnected tournaments by considering their 3-cycles in the following theorem.

Theorem 5.2. *A tournament T is simply disconnected if and only if:*

1. *there exists a non-coned 3-cycle in T ;*
2. *each coned 3-cycle of T is included in a component of T .*

Proof. If the tournament T is simply disconnected, T includes some non-coned 3-cycles, since every non-nullhomotopic 3-cycle is non-coned too.

Moreover each coned 3-cycle C of T must be included in a component of T . Otherwise the projection of C in the simple quotient tournament related to T is a 3-cycle C' of a highly regular tournament and C' is not coned. But this is a contradiction.

The converse is obtained by using induction on the order n ($n \geq 3$) of the tournament T and by following the proof of Theorem 5.1, step 3. □

6. SIMPLY DISCONNECTED SEMICOMPLETE DIGRAPHS

In this section we generalize Theorem 5.1 and 5.2 to semicomplete digraphs (see [16]).

Definition 6.1. Let D be a digraph. A pair of different vertices x, y of D is said to be *symmetric* if there are both arcs $x \rightarrow y$ and $y \rightarrow x$ in D .

Definition 6.2. A digraph D is said to be *semicomplete* if every pair of different vertices of D is joined by at least one arc.

Every tournament is obviously a semicomplete digraph too.

Remark 6.1. Given a complete digraph D_n of order n , there exists at least a tournament T_n of order n , which is a subdigraph of D_n . In fact it is sufficient to delete an arc from each symmetric pairs of D_n .

As in the case of tournaments a semicomplete digraph D can also be endowed with an algebraic structure, since D becomes a groupoid, by putting:

$$(\forall x, y \in D) \quad x \circ y = \begin{cases} x, & \text{if } x \rightarrow y \text{ or } x = y; \\ y, & \text{if } y \rightarrow x \text{ and } x \not\rightarrow y. \end{cases}$$

So we can consider quotients of complete digraphs and we obtain results similar to the ones presented in section 3. The main difference between tournaments and digraphs, in this case, is the fact that for digraphs homomorphisms do not correspond to algebraic homomorphisms as it is so for tournaments.

Definition 6.3. A semicomplete digraph D is called *simply connected* (*simply disconnected*) if the first homotopy group $Q_1(D)$ of D is trivial (non-trivial).

Then we get the following generalization of Theorem 5.1

Theorem 6.1. *A semicomplete digraph D is simply disconnected if and only if its simple quotient is a highly regular tournament.*

Proof. If the simple quotient of D is highly regular, D is simply disconnected by steps 1 and 2 of Theorem 1 of the previous section.

If the simple quotient of D is not highly regular, we can show that D is simply connected by using the following

□

Lemma 6.1. *Let D_n and D'_n be semicomplete digraphs of the same order n such that D'_n is a subdigraph of D_n . If D'_n is simply connected, then D_n is simply connected too.*

We also have the generalization of Theorem 5.2, given by the following theorem.

Theorem 6.2. *A semicomplete digraph D is simply disconnected if and only if:*

1. *there exists a non-coned 3-cycle in D ;*
2. *each symmetric pair and each coned 3-cycle of D are included in a component of D .*

7. FINAL REMARKS

In this last section we shall present some results, showing how this new approach to study digraphs from a homotopical viewpoint has brought some freshness to this area, specially in the case of hamiltonian tournaments.

For instace, taking a closer look at the concept of coned cycles in a tournament T , we will show how this concept relates to some of the homotopical properties of T .

First of all, we recall that for a tournament T the *simpli-*
cial complex K_T associated is such that its 0-simplices are given by the vertices of T , and all other simplices are given as follows: $S = (v_1, \dots, v_n) \in K_T$ if and only if the subtournament $\langle v_1, \dots, v_n \rangle$ induced by the vertices v_1, \dots, v_n is transitive.

Now let a cycle $C : v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_r \rightarrow v_1$ be coned by a vertex v in T ; let us say for instance that $C \rightarrow v$.

We can see that all the subtournaments $S^i = \langle v_i, v_{i+1}, v \rangle$ with v_i, v_{i+1} in C , are all transitive. Therefore $(v_i, v_{i+1}, v) \in K_T$ and hence the cycle C is nullhomotopic.

If C is a non-coned cycle instead, then we do not have this situation. An important application of the concept of non-coned cycle was given by Burzio and Demaria in [9]. We have the following:

Theorem 7.1. *A tournament H_n ($n \geq 5$) is hamiltonian if and only if there exists an m -cycle C , with $3 \leq m \leq n - 2$, which is non-coned in H_n .*

Proof. Let us suppose H_n is hamiltonian. Let v be a neutral vertex of H_n and v_1, v_2 two neutral vertices of $H_n \setminus v$. Let us suppose by contradiction that the two hamiltonian subtournaments $H_n \setminus \{v, v_1\}$ and $H_n \setminus \{v, v_2\}$ are both coned. Since v_1 cannot cone $H_n \setminus \{v, v_1\}$ (otherwise $H_n \setminus \{v, v_1\}$ is not hamiltonian), and v_2 cannot cone $H_n \setminus \{v, v_2\}$ (otherwise $H_n \setminus \{v, v_2\}$ is not hamiltonian), then both $H_n \setminus \{v, v_1\}$ and $H_n \setminus \{v, v_2\}$ are coned by v . Hence v cones $H_n \setminus v$, which is a contradiction since H_n is hamiltonian. Therefore at least one of the two subtournaments $H_n \setminus \{v, v_1\}$ and $H_n \setminus \{v, v_2\}$ is non-coned. In other words, in H_n there exists at least one non-coned $(n - 2)$ -cycle. Conversely, if T_n is not hamiltonian, then its simple quotient is T_2 . Hence every cycle of T_n is included in an e -component, and therefore it is coned. \square

Remark 7.1. Also H_3 and H_4 contain non-coned m -cycles, but in this case the condition $m \leq n - 2$ is not satisfied.

If C is a non-coned cycle of H_n and $v \notin V(C)$, then it is possible to extend C to a cycle through all the vertices of $H_n \setminus v$. This fact motivated Burzio and Demaria to define:

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Definition 7.1. Let H_n be a hamiltonian tournament. A vertex v of H_n is called a *neutral vertex* of H_n if $H_n \setminus v$ is hamiltonian. The number of the neutral vertices of H_n is denote by $\nu(H_n)$.

Remark 7.2. We observe that $\nu(H_n)$ is also the number of hamiltonian subtournaments of order $n - 1$, so we have that $\nu(H_n) \leq n$, for we can have at most n subtournaments of order $n - 1$. On the other hand, in [22] Moon proved that the minimum number of k -cycles, with $3 \leq k \leq n$, in a hamiltonian tournament H_n is equal to $n - k + 1$. Hence we have that $\nu(H_n) \geq 2$, if $n \geq 4$. Therefore, we have that $2 \leq \nu(H_n) \leq n$, for $n \geq 4$.

Definition 7.2. Let C be a non-coned cycle of H_n . The set $P_c = V(H_n) \setminus V(C)$ consists of neutral vertices of H_n , and these are called *poles* of C . A non-coned cycle C of H_n is said to be *minimal* if every cycle C' , such that $V(C') \subset V(C)$, is coned by at least one vertex of H_n . A minimal cycle is said to be *characteristic* if it possesses the shortest length of the minimal cycles. The length of a characteristic cycle is called the *cyclic characteristic* of H_n and it is denoted by $cc(H_n)$. The

difference $n - cc(H_n)$ is called the *cyclic difference* of H_n and is denoted by $cd(H_n)$.

We observe that if C is a characteristic cycle of H_n , then $cd(H_n) = |P_c|$.

Using these definitions and the result given in Theorem 5.5, Burzio and Demaria in [7] gave a classification for the collection \mathcal{H}_n of all the hamiltonian tournaments of order $n \geq 5$, subdividing it in $n - 4$ different classes. Namely, the first class of cyclic characteristic 3 is formed by the tournaments which contain a non-coned 3-cycle; the second one of cyclic characteristic 4 consists of the tournaments which contain a non-coned 4-cycle and whose 3-cycles are all coned. And so on, till the $(n-4)$ th class of cyclic characteristic $(n-2)$ which consists of the tournaments containing a non-coned $(n - 2)$ -cycle and whose cycles with lower length are all coned.

Formally we have the following:

Theorem 7.2. *Let H_n , with $n \geq 5$, be a hamiltonian tournament, then $2 \leq cd(H_n) \leq n - 3$ (or equivalently $3 \leq cc(H_n) \leq n - 2$). Conversely, for every $n \geq 5$ and for every h such that $2 \leq h \leq n - 3$, there exist hamiltonian tournaments H_n with $cd(H_n) = h$.*

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This classification theorem for the hamiltonian tournaments has led to some important recent results, due to the fact this new invariant $cc(H_n)$ (cyclic characteristic) can be obtained in a combinatorial way (just using the adjacency data) and it has some nice properties, like the one given in the next proposition.

Proposition 7.1. *If a tournament T_n is the composition $R_m(S^{(1)}, S^{(2)}, \dots, S^{(m)})$, then $cc(T_n) = cc(R_m)$.*

These results have allowed to obtain the structural characterization of some important classes of hamiltonian tournaments.

Demaria and Gianella defined T_n to be a *normal tournament* if it is hamiltonian and has a unique characteristic cycle. In [14] they thoroughly studied this class of tournaments, which turned out to be very important to obtain structural characterization theorems for other classes of tournaments (see [17, 20, 21]).

We present here some of the most important properties of the normal tournaments.

Definition 7.3. The tournament A_n , with $n \geq 4$, such that $V(A_n) = \{a_1, a_2, \dots, a_n\}$ and $V(A_n) = \{a_i \rightarrow a_j : j < i - 1 \text{ or } j = i + 1\}$, is called the *bineutral tournament of order n* . If $n = 3$, we put A_3 to be the 3-cycle.

In [14] Demaria and Gianella have also shown that a normal tournament H_n has as its characteristic cycle either the 3-cycle A_3 or a bineutral tournament A_k ($n \geq 4$). In the same paper they have proved the following:

Proposition 7.2. *Let H_n be a normal tournament with cyclic characteristic k ($k \geq 3$) and let A_k be its characteristic cycle. A pole z , associated to A_k , must have the following adjacencies with respect to A_k :*

1. $(a_{i+1}, a_{i+2}, \dots, a_k) \rightarrow z \rightarrow (a_1, a_2, \dots, a_i)$ ($1 \leq i \leq k - 1$).
2. $(a_i, a_{i+2}, a_{i+3}, \dots, a_k) \rightarrow z \rightarrow (a_1, \dots, a_{i-1}, a_{i+1})$ ($1 \leq i \leq k - 1$).

Definition 7.4. The pole z is called a *pole of kind i and class 1 or class 2* (and denoted by x_i or y_i) if its adjacencies are given by the previous conditions 1) or 2), respectively.

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The class of the normal tournaments is very important in the study of the hamiltonian tournaments, for instance, the class of the hamiltonian tournaments which have a unique n -cycle, which was characterized by Douglas (see [19]), can now be characterized in a different way as it is shown in the next proposition.

Proposition 7.3. *Let H_n be a hamiltonian tournament with $cc(H_n) = k \geq 3$. H_n is a Douglas tournament if, and only if:*

- 1.1) H_n has as a simple quotient Q_m ($m \geq 5$) such that:
 - a) Q_m is normal;
 - b) the subtournament of the poles in Q_m is transitive;
 - c) the poles of Q_m are all of class 1;
 - d) between two poles x_i and x'_j of Q_m of class 1, the following rules of adjacencies hold $x_i \rightarrow x'_j$ implies $j \leq i + 1$.
- 1.2) H_n can be constructed from Q_m by replacing all the vertices of Q_m , but the vertices a_2, \dots, a_{k-1} of its characteristic cycle A_k , by some transitive tournament.

2) H_n is the composition of a singleton and two transitive tournaments with a 3-cycle.

(See [17])

Later Demaria and Kiihl, using this characterization and the structural characterization of the normal tournaments given in Proposition 6.2, obtained the enumeration of the Douglas tournaments with a convenient variation of the Pascal triangle (see [18]).

Recently Demaria, Guido and others (see [15], [20], [21] and [22]) have used the concept of non-coned cycles in order to approach the reconstruction problem for tournaments. It is known (see [229, 30]) that the reconstruction conjecture fails for tournaments. Then the challenge is to find a characterization (if any) of reconstructable or non-reconstructable tournaments. In this context the reconstruction of combinatorial properties and invariants of tournaments certainly are very useful. In [22] Guido and Kiihl computed the cyclic characteristic of all known tournaments which are non-reconstructable. It is early to say but it seems there might be some direct link between reconstructable hamiltonian tournaments and their cyclic characteristic. In fact, as it is pointed out in [22], no non-reconstructable tournament is known having $cc(H) > 4$.

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